Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME	
CENTRE NUMBER	
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6916036546

FURTHER MATHEMATICS **9231/11**

Paper 1 Further Pure Mathematics 1 **October/November 2020**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **20** pages. Blank pages are indicated.

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1 The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, where a and b are positive constants.

(a) The matrix **M** represents a sequence of two geometrical transformations.

State the type of each transformation, and make clear the order in which they are applied. [2]

Stretch parralel to x-axis, s.c b. shear parralel to x-axis, s.c a.

The unit square in the x-y plane is transformed by **M** onto parallelogram $OPQR$.

(b) Find, in terms of a and b, the matrix which transforms parallelogram $OPQR$ onto the unit square. [2]

$M = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$

$\det(M) = a$

$M^{-1} = \frac{1}{\det(M)}\begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix}$

$= \frac{1}{a}\begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix}$

It is given that the area of $OPQR$ is $2\,\text{cm}^2$ and that the line $x + 3y = 0$ is invariant under the transformation represented by $\mathbf{M}$.

(c) Find the values of a and b. [5]

$a = 2$

$x + 3y = 0$

$y = -\frac{1}{3}x$

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} k \\ -\frac{1}{3}k \end{pmatrix} = \begin{pmatrix} k \\ -\frac{1}{3}k \end{pmatrix}$$

$$\begin{pmatrix} ak - \frac{1}{3}bk \\ -\frac{1}{3}k \end{pmatrix} = \begin{pmatrix} k \\ -\frac{1}{3}k \end{pmatrix}$$

$ak - \frac{1}{3}bk = k$

$2k - \frac{1}{3}bk = k$

$-\frac{1}{3}b = -1$

$b = 3$

2 **(a)** Use standard results from the List of Formulae (MF19) to show that

$$\sum_{r=1}^{n}(7r+1)(7r+8) = an^3 + bn^2 + cn,$$

where a, b and c are constants to be determined. [3]

$= \sum_{r=1}^{n}(49r^2+63r+8)$

$= 49\sum_{r=1}^{n}r^2 + 63\sum_{r=1}^{n}r + \sum_{r=1}^{n}8$

$= 49\left[\frac{1}{6}n(n+1)(2n+1)\right] + 63\left[\frac{1}{2}n(n+1)\right] + 8n$

$= \frac{49}{6}n(2n^2+3n+1) + \frac{63}{2}n(n+1) + 8n$

$= \frac{49}{3}n^3 + \frac{49}{2}n^2 + \frac{49}{6}n + \frac{63}{2}n^2 + \frac{63}{2}n + 8n$

$= \frac{49}{3}n^3 + 56n^2 + \frac{143}{3}n$

(b) Use the method of differences to find $\sum_{r=1}^{n} \frac{1}{(7r+1)(7r+8)}$ in terms of n. [4]

$$\frac{1}{(7r+1)(7r+8)} = \frac{1}{7}\left(\frac{1}{7r+1} - \frac{1}{7r+8}\right)$$

$$\sum_{r=1}^{n} \frac{1}{(7r+1)(7r+8)} = \frac{1}{7}\sum_{r=1}^{n}\left(\frac{1}{7r+1} - \frac{1}{7r+8}\right)$$

$$= \left[\frac{1}{8} - \cancel{\frac{1}{15}}\right.$$
$$\left.\cancel{\frac{1}{15}} - \cancel{\frac{1}{22}}\right]$$

$$\left[\cancel{\frac{1}{7n-6}} - \cancel{\frac{1}{7n+1}}\right.$$
$$\left.\cancel{\frac{1}{7n+1}} - \frac{1}{7n+8}\right]$$

$$= \frac{1}{8} - \frac{1}{7n+8}$$

$$\frac{1}{7}\sum_{r=1}^{n}\left(\frac{1}{(7r+1)} - \frac{1}{(7r+8)}\right) = \frac{1}{7}\left(\frac{1}{8} - \frac{1}{7n+8}\right)$$

(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{(7r+1)(7r+8)}$. [1]

$$\lim_{n\to\infty} \frac{1}{7}\left(\frac{1}{8} - \frac{1}{7n+8}\right)$$

$$= \frac{1}{7}\left(\frac{1}{8} - 0\right)$$

$$= \frac{1}{56}$$

3 The cubic equation $x^3 + cx + 1 = 0$, where c is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are α^3, β^3, γ^3. [3]

let $y = x^3$

$x = y^{1/3}$

$(y^{1/3})^3 + c(y^{\frac{1}{3}}) + 1 = 0$

$y + cy^{1/3} + 1 = 0$

$(y+1) = (-cy^{1/3})$

$(y+1)^3 = (-cy^{1/3})^{1/3}$

$-c^3 y = (y+1)(y^2+2y+1)$

$-c^3 y = y^3 + 2y^2 + y + y^2 + 2y + 1$

$-c^3 y = y^3 + 3y^2 + 3y + 1$

$y^3 + 3y^2 + c^3 y + 3y + 1 = 0$

(b) Show that $\alpha^6 + \beta^6 + \gamma^6 = 3 - 2c^3$. [3]

$y^3 + 3y^2 + (c^2 + 3)y + 1 = 0$

$\alpha^3 + \beta^3 + \gamma^3 = -3$

$\alpha^3\beta^3 + \alpha^3\gamma^3 + \beta^3\gamma^3 = c^3 + 3$

$\gamma^3\beta^3\gamma^3 = -1$

$\alpha^6 + \beta^6 + \gamma^6 = (\alpha^3 + \beta^3 + \gamma^3)^2 - 2(\alpha^3\beta^3 + \alpha^3\gamma^3 + \beta^3\gamma^3)$

$= (-3)^2 - 2(c^3 + 3)$

$= 9 - 2c^3 - 6$

$= 3 - 2c^3$

(c) Find the real value of c for which the matrix $\begin{pmatrix} 1 & \alpha^3 & \beta^3 \\ \alpha^3 & 1 & \gamma^3 \\ \beta^3 & \gamma^3 & 1 \end{pmatrix}$ is singular. [5]

$= 1\begin{vmatrix} 1 & \gamma^3 \\ \gamma^3 & 1 \end{vmatrix} - \alpha^3\begin{vmatrix} \alpha^3 & \gamma^3 \\ \beta^3 & 1 \end{vmatrix} + \beta^3\begin{vmatrix} \alpha^3 & 1 \\ \beta^3 & \gamma^3 \end{vmatrix}$

$= 1(1-\gamma^6) - \alpha^3(\alpha^3 - \beta^3\gamma^3) + \beta^3(\alpha^3\gamma^3 - \beta^3)$

$= 1 - \gamma^6 - \alpha^6 + \alpha^3\beta^3\gamma^3 + \alpha^3\beta^3\gamma^3 - \beta^6$

$= 1 - (\alpha^6 + \beta^6 + \gamma^6) + 2(\alpha^3\beta^3\gamma^3)$

$= 1 - (3 - 2c^3) + 2(-1)$

$= 1 - 3 + 2c^3 - 2$

$= -4 + 2c^3$

$-4 + 2c^3 = 0$

$2c^3 = 4$

$c^3 = 2$

$c = \sqrt[3]{2}$

4 The points A, B, C have position vectors

$$-\mathbf{i}+\mathbf{j}+2\mathbf{k}, \quad -2\mathbf{i}-\mathbf{j}, \quad 2\mathbf{i}+2\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (-2i - j) - (-i + j + 2k)$

$= -i - 2j - 2k$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= (2i + 2k) - (-i + j + 2k)$

$= 3i - j$

$\vec{n} = \begin{vmatrix} i & j & k \\ -1 & -2 & -2 \\ 3 & -1 & 0 \end{vmatrix}$

$= -2i - 6j + 7k$

$d = a \cdot n$

$= \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -6 \\ 7 \end{pmatrix}$

$2 - 6 + 14 = 0$

$= 10$

$r \cdot (-2i - 6j + 7k) = 10$

$-2x - 6y + 7y = 10$

(b) Find the perpendicular distance from O to the plane ABC. [2]

$$\frac{d}{|\vec{n}|} = \frac{10}{\sqrt{2^2+6^2+7^2}}$$

$$= \frac{10}{\sqrt{89}}$$

(c) Find the acute angle between the planes OAB and ABC. [4]

$$\vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1||\vec{n}_2|\cos\theta$$

$$\vec{n}_1 = \begin{vmatrix} i & j & k \\ -1 & 1 & 2 \\ -2 & -1 & 0 \end{vmatrix}$$

$$= 2i - 4j + 3k$$

$$\begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -6 \\ 7 \end{pmatrix} = \sqrt{2^2+4^2+3^2} \cdot \sqrt{2^2+6^2+7^2} \cos\theta$$

$$-4+24+21 = \sqrt{29} \cdot \sqrt{89} \cos\theta$$

$$\cos\theta = \frac{41}{\sqrt{29}\sqrt{89}}$$

$$\theta = \cos^{-1}\left(\frac{41}{\sqrt{29}\sqrt{89}}\right)$$

$$\theta = 36.2°$$

5 Prove by mathematical induction that, for every positive integer n,

$$\frac{d^{2n-1}}{dx^{2n-1}}(x\sin x) = (-1)^{n-1}\left(x\cos x + (2n-1)\sin x\right).$$ [7]

Prove for $n = 1$:

$\frac{d}{dx}\left(x\sin x\right) = \sin x + x\cos x$

RHS: $(-1)^0(x\cos x + \sin x)$

$= x\cos x \sin x$

$x\cos x\sin x = x\cos x\sin x$

Suppose for $n = k$

$\frac{d^{(2k-1)}}{dx^{(2k-1)}}\left(x\sin x\right) = (-1)^{k-1}\left(x\cos x + (2k-1)\sin x\right)$

Prove for $n = k+1$

$\frac{d^{2k+1}}{dx^{2k+1}}(x\sin x) = \frac{d}{dx}\left(\frac{d^{2k}}{dx^{2k}} x\sin x\right)$

$\frac{d^{2k}}{dx^{2k}}(x\sin x) = \frac{d}{dx}\left(\frac{d^{2k-1}}{dx^{2k-1}} x\sin x\right)$

$= \frac{d}{dx}\left((-1)^{k-1}(x\cos x + (2k-1)\sin x\right)$

$= (-1)^{k-1}(\cos x - x\sin x + (2k-1)\cos x)$

$= (-1)^{k-1}(-x\sin x + 2k\cos x)$

$$\frac{d^{2k+1}}{dx^{2k+1}}(x\sin x) = \frac{d}{dx}\left[(-1)^{k-1}(-x\sin x) + 2k(\cos x)\right]$$

$$= (-1)^{k-1}(-\sin x - x\cos x - 2k\sin x)$$

$$= (-1)^{k}(x\cos x + (2k+1)\sin x)$$

$\therefore$ true for $n=k+1$ hence it is true for all positive integers of n.

6 The curve C has equation $y = \dfrac{x^2 + x - 1}{x - 1}$.

(a) Find the equations of the asymptotes of C. [3]

$x - 1 = 0$

$x = 0$

$$\begin{array}{r} x+2 \\ x-1 \overline{\big) x^2 + x - 1} \\ -\ \underline{x^2 + x} \\ 2x - 1 \\ -\ \underline{2x + 2} \\ 1 \end{array}$$

$y = x + 2$

V_{asy}: $x = 0$

O_{asy}: $y = x + 2$

(b) Show that there is no point on C for which $1 < y < 5$. [4]

$y(x-1) = x^2 + x - 1$

$xy - y = x^2 + x - 1$

$x^2 + x - xy + y - 1 = 0$

$\underset{a}{1}x^2 + x\underset{b}{(1-y)} + \underset{c}{y-1} = 0$

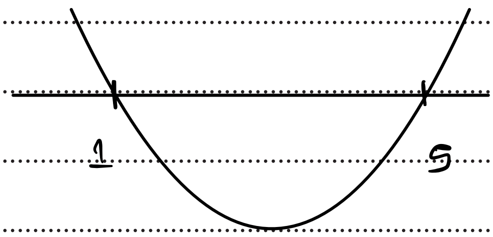
$b^2 - 4ac < 0$

$(1-y)^2 - 4(1)(y-1) = 0$

$1 - 2y + y^2 - 4y + 4 < 0$

$y^2 - 6y + 5 < 0$

$(y-5)(y-1) < 0$



$\therefore 1 < y < 5$

(c) Find the coordinates of the intersections of C with the axes, and sketch C. [3]

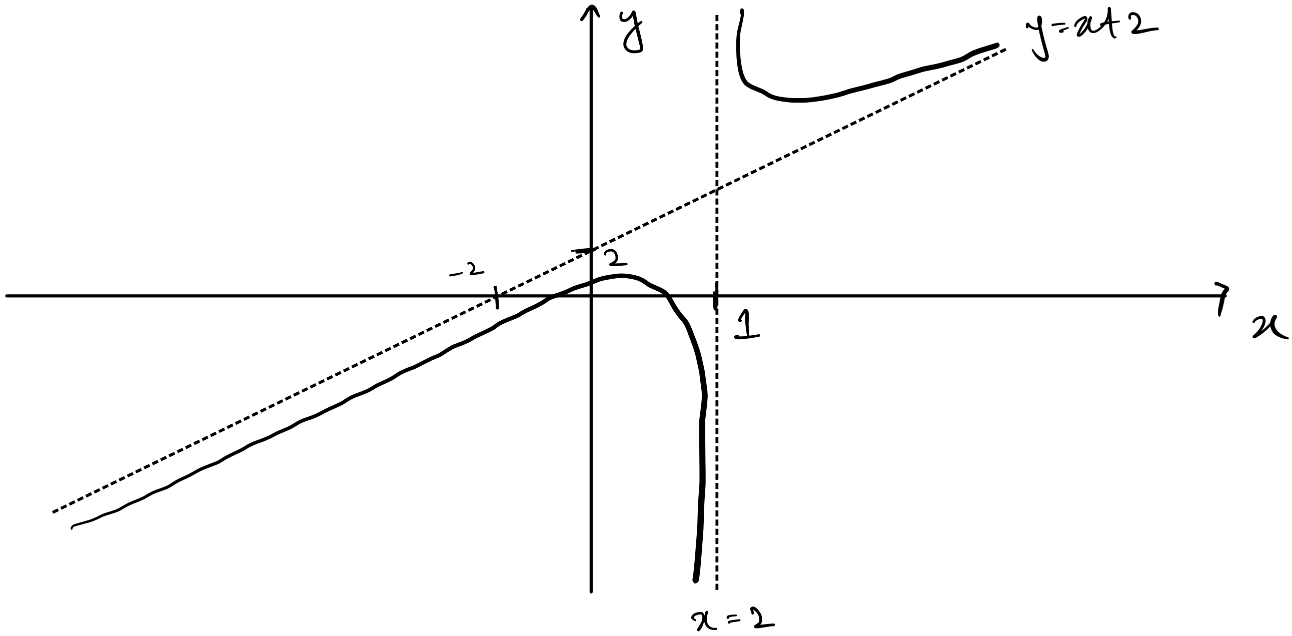
y-intercept $= (0,1)$

x-intercept

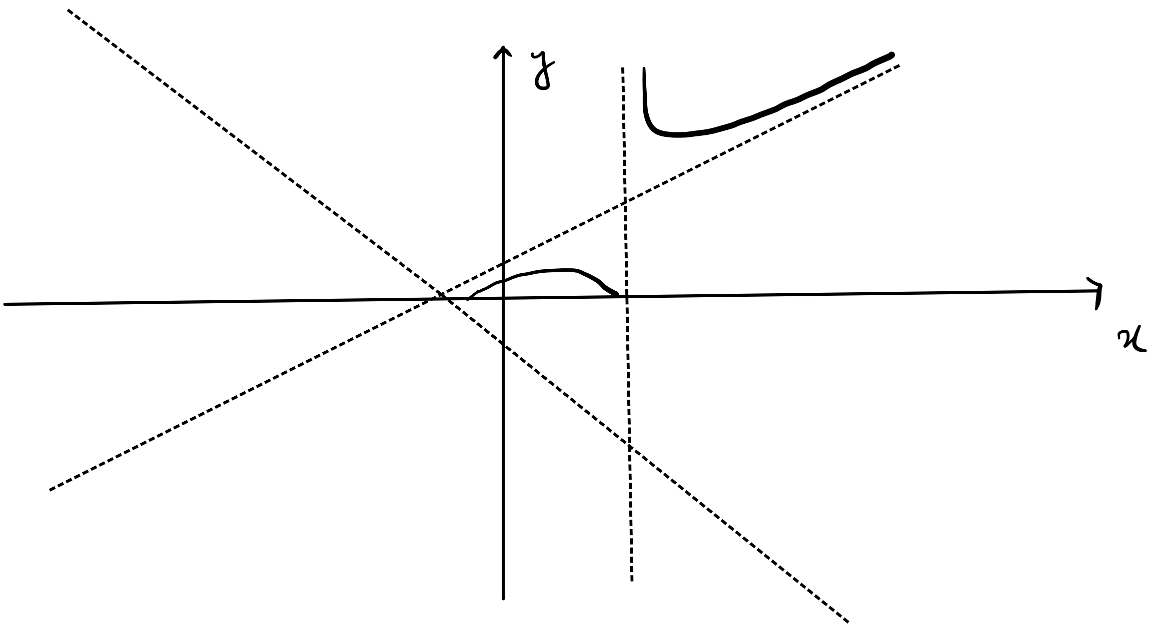
$x^2+x-1=0$

$x = \frac{-1 \pm \sqrt{5}}{2}$

$\left(\frac{-1+\sqrt{5}}{2}, 0\right)$ & $\left(\frac{-1-\sqrt{5}}{2}, 0\right)$



(d) Sketch the curve with equation $y = \left|\frac{x^2+x-1}{x-1}\right|$. [2]



Additional Page

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