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Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

7435775880

FURTHER MATHEMATICS **9231/11**

Paper 1 Further Pure Mathematics 1 **May/June 2020**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Blank pages are indicated.

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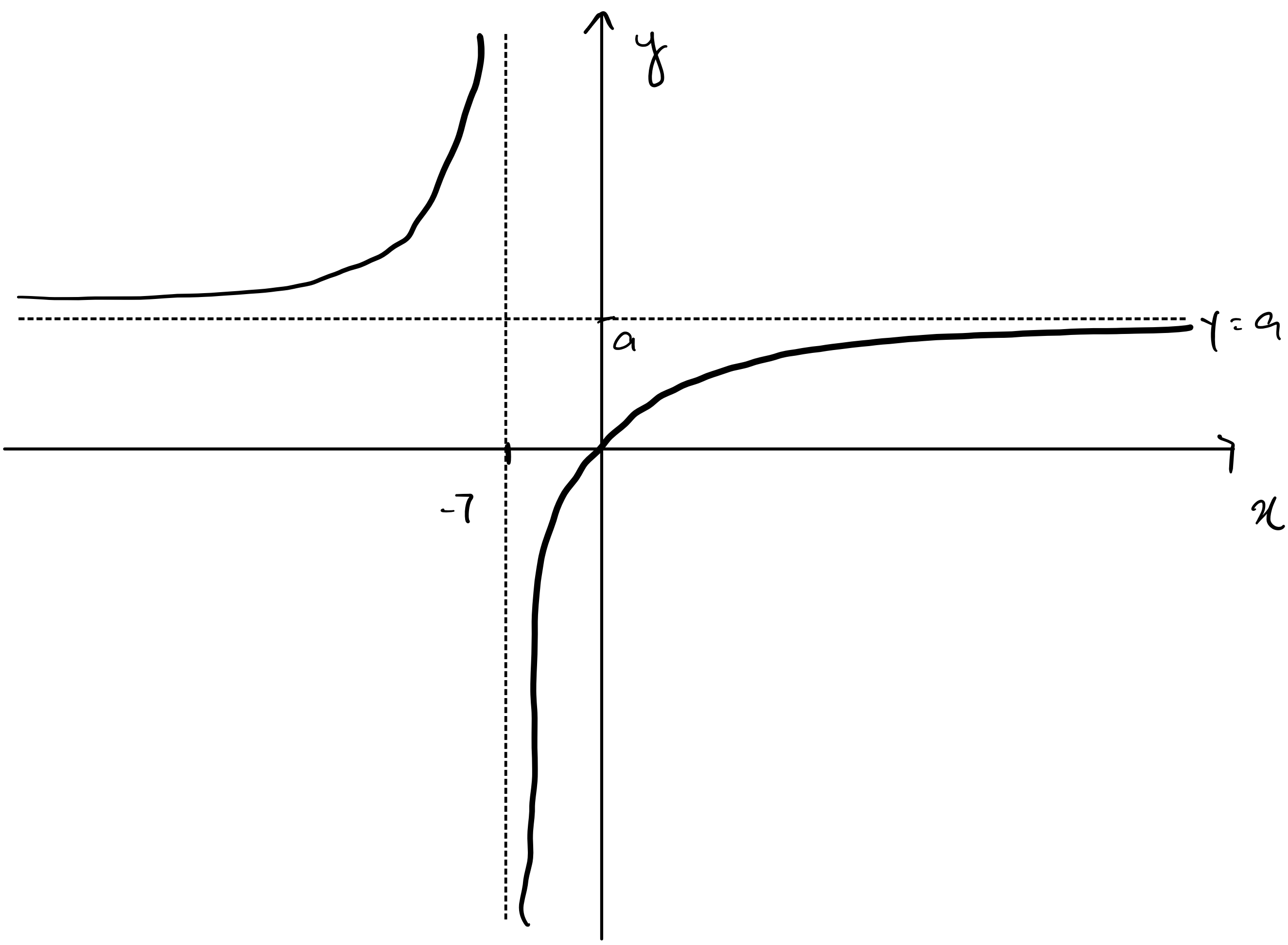
1 Let a be a positive constant.

(a) Sketch the curve with equation $y = \dfrac{ax}{x+7}$. [2]

V_{asy} : $x+7=0$

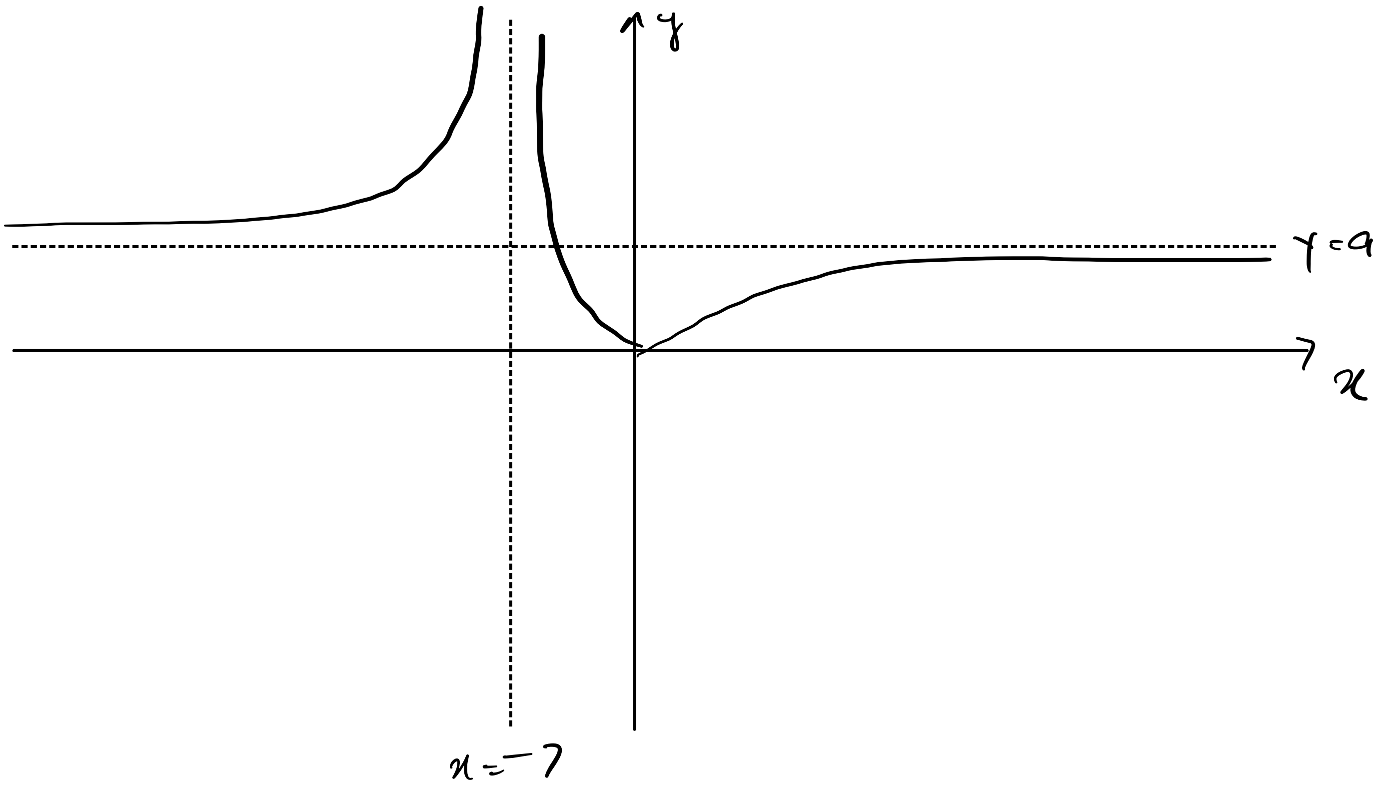
$x=-7$

H_{asy} : $y=a$



(b) Sketch the curve with equation $y = \left|\frac{ax}{x+7}\right|$ and find the set of values of x for which $\left|\frac{ax}{x+7}\right| > \frac{a}{2}$.

[4]



$\frac{ax}{x+7} = \frac{a}{2}$ | $-\frac{ax}{x+7} = \frac{a}{2}$

$2x = x+7$ | $-2x = x+7$

$x = 7$ | $x = -\frac{7}{3}$

$y > 7$, $-7 < x < -\frac{7}{3}$, $x < -7$

2 The cubic equation $6x^3 + px^2 - 3x - 5 = 0$, where p is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are α^2, β^2, γ^2. [3]

let $y = x^2$

$x = y^{1/2}$

$6(y^{\frac{1}{2}})^3 + p(y^{\frac{1}{2}})^2 - 3(y^{\frac{1}{2}}) - 5 = 0$

$6y^{3/2} + py^{1/2} - 3y^{\frac{1}{2}} - 5 = 0$

$6y \cdot y^{1/2} - 3y^{1/2} = 5 - py$

$y^{1/2}(6y-3) = 5-py$

$[y^{1/2}(6y-3)]^2 = (5-py)^2$

$y(36y^2 - 36y + 9) = 25 - 10py + p^2y^2$

$36y^3 - (36+p^2)y^2 + (9+10p)y - 25 = 0$

(b) It is given that $\alpha^2 + \beta^2 + \gamma^2 = 2(\alpha + \beta + \gamma)$.

(i) Find the value of p. [3]

$\frac{36+p^2}{36} = 2\left(\frac{-p}{6}\right)$

$3(36+p^2) = -36p$

$36 + p^2 = -12p$

$p^2 + 12p + 36 = 0$

$p = -6$

(ii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. [2]

$6x^3 - 6x^2 - 3x - 5 = 0$

$6(\alpha^3+\beta^3+\gamma^3) = 6(\alpha^2+\beta^2+\gamma^2) + 3(\alpha+\beta+\gamma) + 15$

$= 6\left(\frac{36+36}{36}\right) + 3\left(\frac{6}{6}\right) + 15$

$6(\alpha^3+\beta^3+\gamma^3) = 12+3+15$

$6(\alpha^3+\beta^3+\gamma^3) = 30$

$\alpha^3+\beta^3+\gamma^3 = 5$

3 The curve C has equation $y = \frac{x^2}{2x+1}$.

(a) Find the equations of the asymptotes of C. [3]

$2x+1=0$

$2x=-1$

$x=-\frac{1}{2}$

$$2x+1 \overline{\big) \; x^2 \;} \quad \frac{1}{2}x-\frac{1}{4}$$

$-x^2+\frac{1}{2}x$

$-\frac{1}{2}x$

$+\frac{1}{2}x+\frac{1}{4}$

$\frac{1}{4}$

$V_{asy}: \; x=-\frac{1}{2}$

$O_{asy}: \; y=\frac{1}{2}x-\frac{1}{4}$

(b) Find the coordinates of the stationary points on C. [3]

$y=\frac{x^2}{2x+1}$

$\frac{2x(2x+1)-2x^2}{(2x+1)^2}=0$

$2x^2+2x=0$

$2x(x+1)=0$

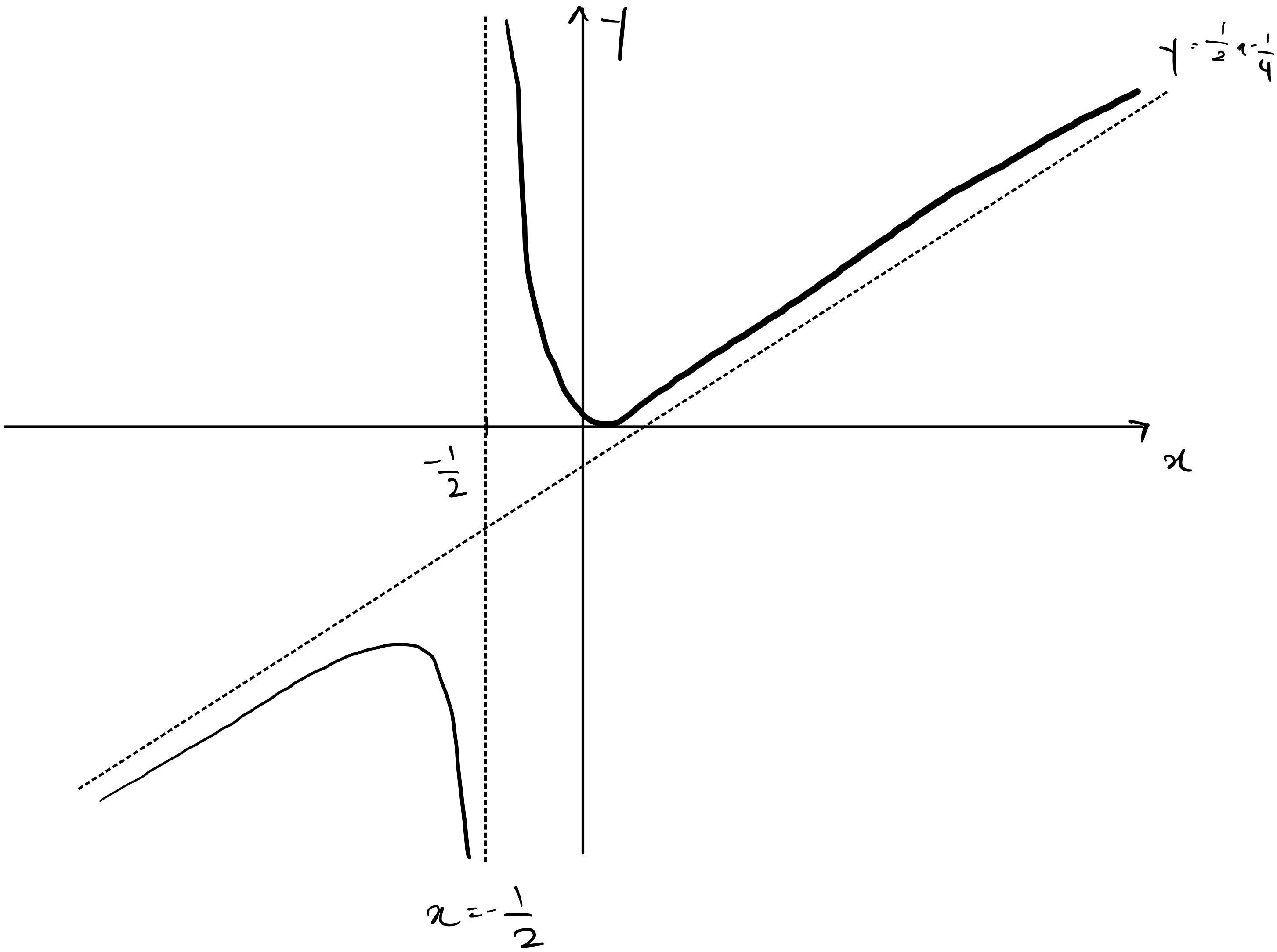
$x=0$

$x=-1$

$(0,0)$

$(-1,-1)$

(c) Sketch *C*. [3]



5 The lines l_1 and l_2 have equations $\mathbf{r} = 3\mathbf{i} + 3\mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} + 4\mathbf{k})$ and $\mathbf{r} = 3\mathbf{i} - 5\mathbf{j} - 6\mathbf{k} + \mu(5\mathbf{j} + 6\mathbf{k})$ respectively.

(a) Find the shortest distance between l_1 and l_2. [5]

$$l_1: r = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}$$

$$l_2: r = \begin{pmatrix} 3 \\ -5 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}$$

$$a_2 - a_1 = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ -5 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 5 \\ 9 \end{pmatrix}$$

$$b_1 \times b_2 = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} \times \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}$$

$$= i(24-20) - j(6-0) + k(5-0)$$

$$= i(4) + j(-6) + k(5)$$

$$|b_1 \times b_2| = \sqrt{4^2 + 6^2 + 5^2}$$

$$= \sqrt{77}$$

$$d = \frac{1}{\sqrt{77}} \left| \begin{pmatrix} 0 \\ 5 \\ 9 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -6 \\ 5 \end{pmatrix} \right|$$

$$= \frac{1}{\sqrt{77}} \left| 0 - 30 + 45 \right|$$

$$= \frac{15}{\sqrt{77}}$$

The plane Π contains l_1 and is parallel to the vector $\mathbf{i}+\mathbf{k}$.

(b) Find the equation of Π, giving your answer in the form $ax+by+cz=d$. [4]

$r \cdot \vec{n} = d$

$\vec{n} = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$= i(4-0) - j(1-4) + k(0-4)$

$= i(4) - j(-3) + k(-4)$

$= i(4) + j(3) + k(-4)$

$= \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \\ -4 \end{pmatrix}$

$= 12 + 0 - 12$

$= 0$

$4x + 3y - 4z = 0$

(c) Find the acute angle between l_2 and Π. [3]

l_1: $r = \begin{pmatrix} 3 \\ -5 \\ -6 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}$

$\sin\theta = \dfrac{\begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \\ -4 \end{pmatrix}}{\sqrt{61}\ \sqrt{41}}$

$\sin\theta = \dfrac{9}{\sqrt{61+41}}$

$\theta = 10.4°$

6 Let $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$.

(a) The transformation in the x-y plane represented by $\mathbf{A}^{-1}$ transforms a triangle of area $30\,\text{cm}^2$ into a triangle of area $d\,\text{cm}^2$.

Find the value of d. [3]

$\det(A) \times d = 30$

$2d = 30$

$d = 15$

(b) Prove by mathematical induction that, for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}.$$ [5]

For $n = 1$:

$A^1 = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$

Suppose for $n = k$:

$A^k = \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix}$

Prove for $n = k+1$:

$A^{k+1} = A^k \cdot A^1$

$= \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 2^{k+1} & 0 \\ 2^{k+1} - 1 & 0 \end{pmatrix}$ $\therefore$ true for $n = k+1$ hence true for all positive values of n.

(c) The line $y = 2x$ is invariant under the transformation in the x-y plane represented by $\mathbf{A}^n\mathbf{B}$, where $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 33 & 0 \end{pmatrix}$.

Find the value of n. [5]

$A^n B = \begin{pmatrix} 2^n & 0 \\ 2^n-1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 33 & 0 \end{pmatrix}$

$= \begin{pmatrix} 2^n & 0 \\ 2^n+32 & 0 \end{pmatrix}$

$\begin{pmatrix} 2^n & 0 \\ 2^n+32 & 0 \end{pmatrix}\begin{pmatrix} k \\ 2k \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$

$\begin{pmatrix} 2^n k \\ k(2^n+32) \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$

$x_1 = 2^n k$

$y_1 = k(2^n+32)$

$k = \dfrac{x}{2^n}$

$y_1 = \dfrac{x}{2^n}(2^n+32)$

$y_1 = \left(\dfrac{2^n+32}{2^n}\right)x$

$\dfrac{2^n+32}{2^n} = 2$

$2^n+32 = 2\cdot 2^n$

$2^n = 32$

$n = 5$

7 The curve C_1 has polar equation $r = \theta \cos\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) The point on C_1 furthest from the line $\theta = \frac{1}{2}\pi$ is denoted by P. Show that, at P,

$$2\theta \tan\theta - 1 = 0$$

and verify that this equation has a root between 0.6 and 0.7. [5]

$x = r\cos\theta$

$= \theta\cos\theta \cdot \cos\theta$

$= \theta\cos^2\theta$

$\frac{dx}{d\theta} = 0$

$= \cos^2(\theta) - 2\theta\cos\theta\sin\theta$

$\cos^2\theta - 2\theta\cos\theta\sin\theta = 0$

$\cos\theta\,(\cos\theta - 2\theta\sin\theta) = 0$

$\cos\theta - 2\theta\sin\theta = 0$

$1 - 2\theta\tan\theta = 0$

$2\theta\tan\theta - 1 = 0$

For 0.6: $2\theta\tan\theta - 1 = -0.179$

For 0.7: $2\theta\tan\theta - 1 = 0.179$

$\therefore$ sign change means root between 0.6 & 0.7

The curve C_2 has polar equation $r = \theta\sin\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. The curves C_1 and C_2 intersect at the pole, denoted by O, and at another point Q.

(b) Find the polar coordinates of Q, giving your answers in exact form. [2]

$\theta\cos\theta = \theta\sin\theta$

$1 = \tan\theta$

$\tan\theta = 1$

$\theta = \tan^{-1}(1)$

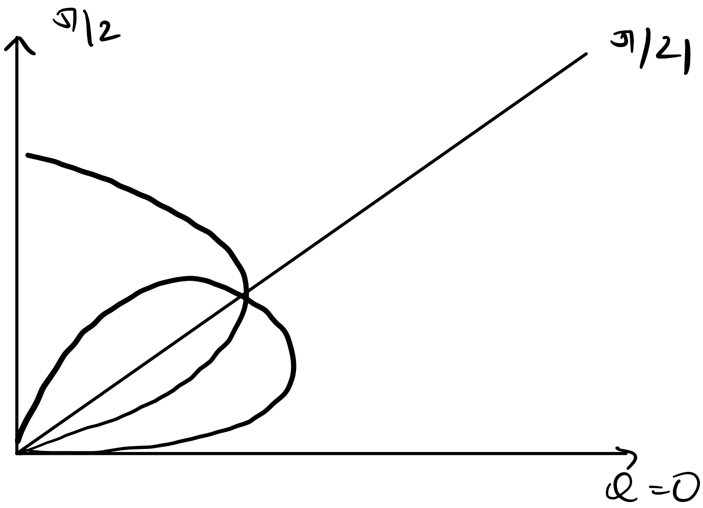
$\theta = \frac{\pi}{4}$

$r = \frac{\pi}{4}\sin\left(\frac{\pi}{4}\right)$

$= \frac{\pi \cdot \sqrt{2}}{8}$

$\left(\frac{\pi\sqrt{2}}{8}, \frac{\pi}{4}\right)$

(c) Sketch C_1 and C_2 on the same diagram. [3]



(d) Find, in terms of π, the area of the region bounded by the arc OQ of C_1 and the arc OQ of C_2. [7]

$$= \frac{1}{2}\int_0^{\pi/4} [\theta\cos\theta]^2 - (\theta\sin\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \theta^2(\cos^2\theta - \sin^2\theta) \, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \theta^2 \cos 2\theta$$

$u = \theta^2$

$\frac{dv}{d\theta} = \cos 2\theta$

$$= uv - \int v \frac{du}{dx}$$

$$= \frac{1}{2}\theta^2 \sin 2\theta - \int \theta \sin 2\theta \, d\theta$$

$u = \theta$

$\frac{dv}{dx} = \sin 2\theta$

$$= -\frac{1}{2}\theta\cos 2\theta + \int \frac{1}{2}\cos 2\theta \, d\theta$$

$$= \frac{1}{2}\theta\cos 2\theta + \frac{1}{4}\sin 2\theta$$

$$= \frac{1}{2}\left|\frac{1}{2}\theta^2\sin 2\theta + \frac{1}{2}\cos 2\theta - \frac{1}{4}\sin 2\theta\right|_0^{\pi/4}$$

$$= \frac{\pi^2}{64} - \frac{1}{8}$$

Additional Page

If you use the following lined page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.