Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

5983925337

FURTHER MATHEMATICS **9231/12**

Paper 1 Further Pure Mathematics 1 **October/November 2021**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Any blank pages are indicated.

1 **(a)** Give full details of the geometrical transformation in the x-y plane represented by the matrix $\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$. [1]

Enlargement scale factor 6

Let $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 2 & 2 \end{pmatrix}$.

(b) The triangle DEF in the x-y plane is transformed by $\mathbf{A}$ onto triangle PQR.

Given that the area of triangle DEF is $13\,\text{cm}^2$, find the area of triangle PQR. [2]

$\det(A) = 6 - 8 = -2$

$\boxed{\text{Area} = 2(13) = 26\,\text{cm}^2}$

(c) Find the matrix $\mathbf{B}$ such that $\mathbf{AB} = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$. [2]

$A^{-1} = -\frac{1}{2}\begin{pmatrix} 2 & -4 \\ -2 & 3 \end{pmatrix}$

$B = -3\begin{pmatrix} 2 & -4 \\ -2 & 3 \end{pmatrix}$

$\boxed{B = \begin{pmatrix} -6 & 12 \\ 6 & 9 \end{pmatrix}}$

(d) Show that the origin is the only invariant point of the transformation in the x-y plane represented by $\mathbf{A}$. [4]

$\begin{pmatrix} 3 & 4 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x+4y \\ 2x+2y \end{pmatrix}$

$3x+4y = x \Rightarrow 2x+4y=0$

$2x+2y = y \Rightarrow 2x+y=0$

$y=-2x$ $\qquad$ $y=0$

$2x+4(-2x)=0$

$-6x=0$

$x=0$

$\boxed{(0,0)}$

2 It is given that $y = xe^{ax}$, where a is a constant.

Prove by mathematical induction that, for all positive integers n,

$$\frac{d^n y}{dx^n} = \left(a^n x + na^{n-1}\right)e^{ax}.$$ [6]

for $n=1$

$\frac{dy}{dx}(xe^{ax}) = axe^{ax} + e^{ax} = (ax+)e^{ax}$ (LHS)

$(ax + 1a^{1-1})e^{ax} = (ax+1)e^{ax}$ (RHS)

LHS = RHS so true

Assume true for $n=k$

$\frac{d^k y}{dx^k}(xe^{ax}) = (a^k x + ka^{k-1})e^{ax}.$

Prove for $n=k+1$

$\frac{d^{k+1}y}{dx^{k+1}}(xe^{ax}) = \frac{d}{dx}\left[\frac{d^k y}{dx^k}(xe^{ax})\right]$

$= \frac{d}{dx}\left[(a^k x + ka^{k-1})e^{ax}\right]$

$= a(a^k x + ka^{k-1})e^{ax} + e^{ax}(a^k)$

$= (a^{k+1} + (k+1)a^k)e^{ax}$

Hence induction is complete.

3 Let $S_n = \sum_{r=1}^{n} \ln \frac{r(r+2)}{(r+1)^2}$.

(a) Using the method of differences, or otherwise, show that $S_n = \ln \frac{n+2}{2(n+1)}$. [4]

$S_n = \sum_{r=1}^{n} \ln r - 2\ln(r+1) + \ln(r+2)$

$\ln 1 - 2\ln 2 + \ln 3$

$\ln 2 - 2\ln 3 + \ln 4$

$\ln 3 - 2\ln 4 + \ln 5$

$\vdots \qquad \vdots$

$\ln(n-1) - 2\ln n + \ln(n+1)$

$\ln n - 2\ln(n+1) + \ln(n+2)$

$-\ln 2 - \ln(n+1) + \ln(n+2)$

$\Rightarrow \boxed{\ln \frac{(n+2)}{2(n+1)}}$

Let $S = \sum_{r=1}^{\infty} \ln \frac{r(r+2)}{(r+1)^2}$.

(b) Find the least value of n such that $S_n - S < 0.01$. [3]

$S = -\ln 2$

$S_n - S = \ln\left(\frac{n+2}{n+1}\right) < 0.01$

$n+2 < e^{0.01}(n+1)$

$\boxed{\text{least is } n = 99}$

4 The cubic equation $x^3 + 2x^2 + 3x + 3 = 0$ has roots α, β, γ.

(a) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [2]

$\alpha^2+\beta^2+\gamma^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$

$= (-2)^2 - 2(3)$

$= 4 - 6$

$\boxed{\alpha^2+\beta^2+\gamma^2 = -2}$

(b) Show that $\alpha^3 + \beta^3 + \gamma^3 = 1$. [2]

$\alpha^3+\beta^3+\gamma^3 = -2(-2) - 3(-2) - 3(3)$

$\boxed{\alpha^3+\beta^3+\gamma^3 = 1}$

(c) Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{n}\left((\alpha+r)^3+(\beta+r)^3+(\gamma+r)^3\right)=n+\tfrac{1}{4}n(n+1)\left(an^2+bn+c\right),$$

where a, b and c are constants to be determined. [6]

$(\alpha+r)^3 = \alpha^3 + 3\alpha^2 r + 3\alpha r^2 + r^3$

$(\beta+r)^3 = \beta^3 + 3\beta^2 r + 3\beta r^2 + r^3$

$(\gamma+r)^3 = \gamma^3 + 3\gamma^2 r + 3\gamma r^2 + r^3$

$\sum_{r=1}^{n}\left(1+3(-2)r+3(-2)r^2+3r^3\right) = \sum_{r=1}^{n}\left(1-6r-6r^2+3r^3\right)$

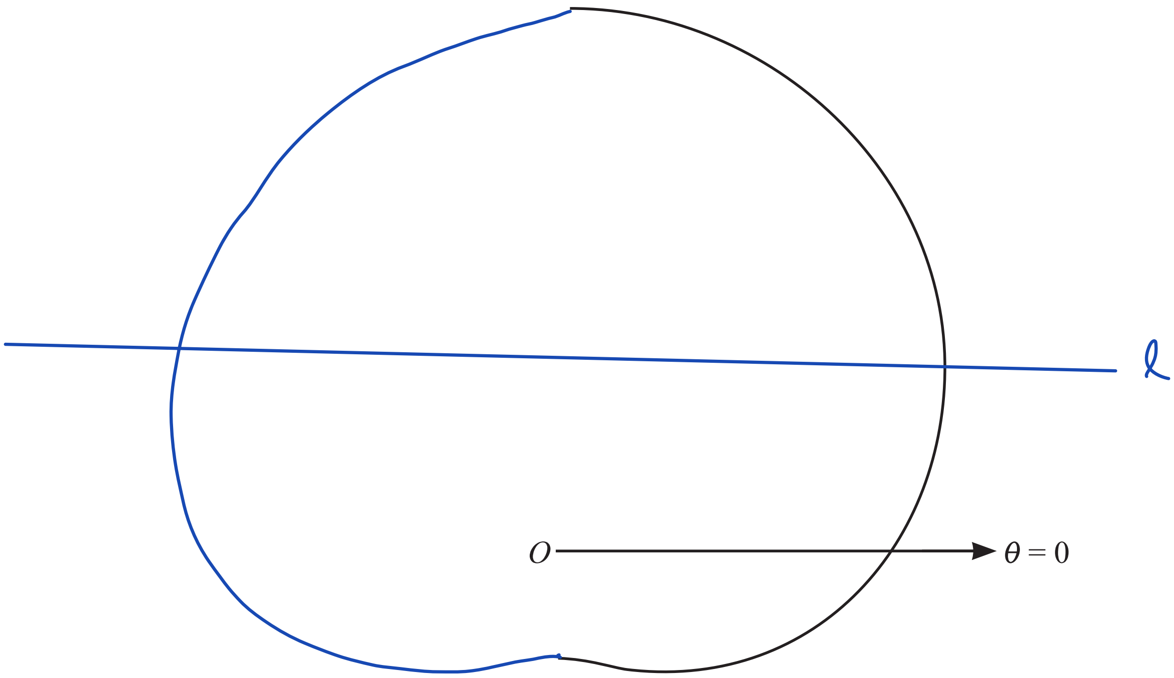
$n - 3n(n+1) - n(n+1)(2n+1) + \frac{3}{4}n^2(n+1)^2$

$n + \frac{1}{4}n(n+1)\left(-12-4(2n+1)+3n(n+1)\right)$

$\boxed{n + \frac{1}{4}n(n+1)(3n^2-5n-16)}$

5 The curve C has polar equation $r = 3 + 2\sin\theta$, for $-\pi < \theta \leqslant \pi$.

(a) The diagram shows part of C. Sketch the rest of C on the diagram. [1]



The straight line l has polar equation $r\sin\theta = 2$.

(b) Add l to the diagram in part **(a)** and find the polar coordinates of the points of intersection of C and l. [5]

$2 = 3\sin\theta + 2\sin^2\theta$

$2\sin^2\theta + 3\sin\theta - 2 = 0$

$2\sin^2\theta + 4\sin\theta - \sin\theta - 2 = 0$

$2\sin\theta(\sin\theta + 2) - 1(\sin\theta + 2) = 0$

$\sin\theta = 1/2$ $\quad\quad$ $\sin\theta = -2$

$\times$ (NOT POSSIBLE

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$\boxed{(4, \frac{\pi}{6}), (4, \frac{5\pi}{6})}$

(c) The region R is enclosed by C and l, and contains the pole.

Find the area of R, giving your answer in exact form. [6]

$$2 \times \frac{1}{2} \int_{-\pi/2}^{\pi/6} (3 + 2\sin\theta)^2 \, d\theta$$

$$\int_{-\pi/2}^{\pi/6} 9 + 12\sin\theta + 2(1 - \cos 2\theta) \, d\theta$$

$$\int_{-\pi/2}^{\pi/6} 9 + 12\sin\theta + 2 - 2\cos 2\theta \quad d\theta$$

$$\int_{-\pi/2}^{\pi/6} 11 + 12\sin\theta - 2\cos 2\theta \quad d\theta$$

$$\left[11\theta - 12\cos\theta - \sin 2\theta\right]_{-\pi/2}^{\pi/6} = \frac{22}{3}\pi - \frac{13}{2}\sqrt{3}$$

$$\frac{22}{3}\pi - \frac{13}{2}\sqrt{3} + 2\left(4\cos\frac{\pi}{6}\right)$$

$$\boxed{\frac{22}{3}\pi - \frac{5}{2}\sqrt{3}}$$

6 The curve C has equation $y = \dfrac{x^2}{x-3}$.

(a) Find the equations of the asymptotes of C. [3]

$\boxed{x=3}$

$y = x+3 + \dfrac{9}{x-3}$

$\boxed{y=x+3}$

(b) Show that there is no point on C for which $0 < y < 12$. [4]

$yx - 3y = x^2$

$x^2 - yx + 3y = 0$

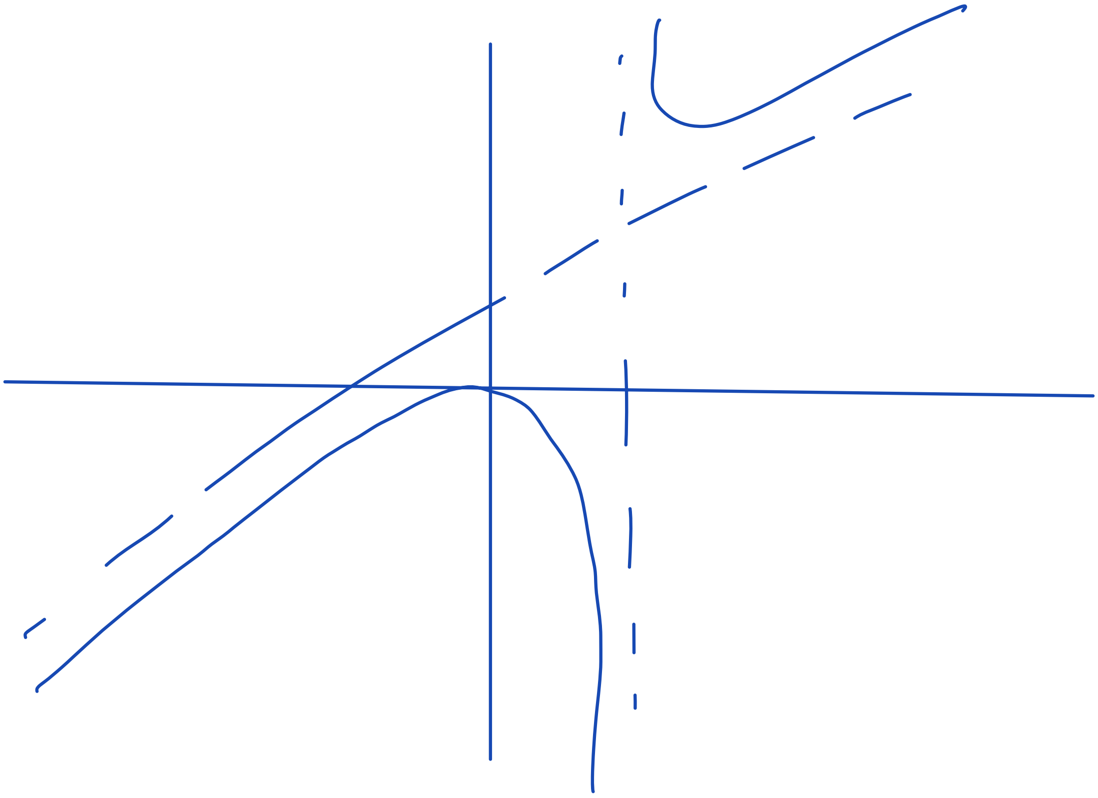
$b^2 - 4ac < 0$

$y^2 - 12y < 0$

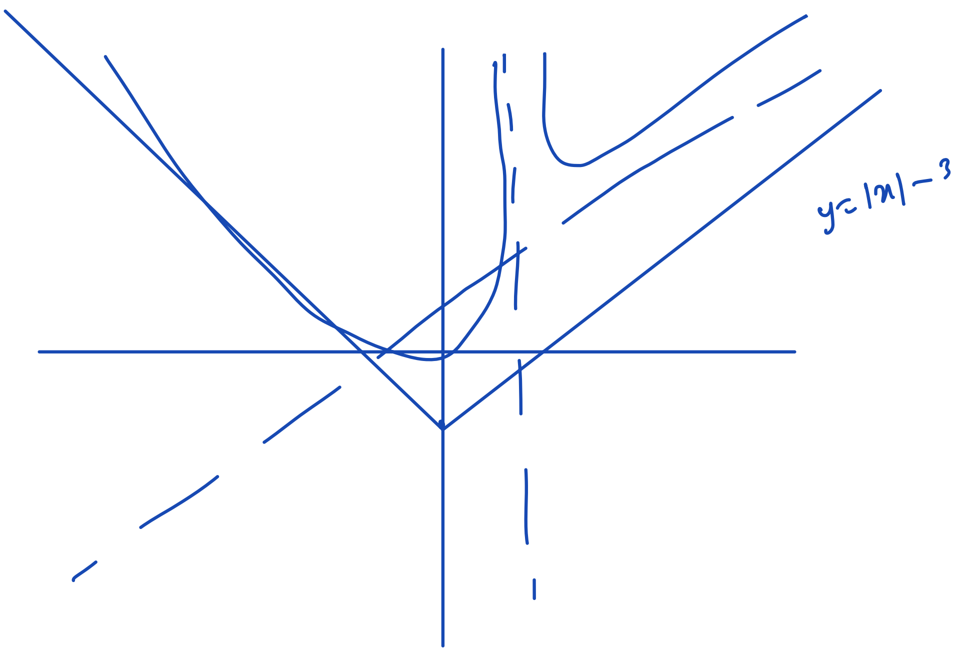
$y(y-12) < 0$

$\boxed{0 < y < 12}$

(c) Sketch C. [2]



(d) **(i)** Sketch the graphs of $y = \left|\frac{x^2}{x-3}\right|$ and $y = |x| - 3$ on a single diagram, stating the coordinates of the intersections with the axes. [4]



(ii) Use your sketch to find the set of values of c for which $\left|\frac{x^2}{x-3}\right| \leqslant |x| + c$ has no solution. [1]

$\boxed{c \leq -3}$

....

....

7 The points A, B, C have position vectors

$$2\mathbf{i}+2\mathbf{j}, \quad -\mathbf{j}+\mathbf{k} \quad \text{and} \quad 2\mathbf{i}+\mathbf{j}-7\mathbf{k}$$

respectively, relative to the origin O.

(a) Find an equation of the plane OAB, giving your answer in the form $\mathbf{r}.\mathbf{n} = p$. [3]

$$\vec{n} = \begin{vmatrix} i & j & k \\ 2 & 2 & 0 \\ 0 & -1 & 1 \end{vmatrix} = \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$\boxed{r \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = 0}$$

The plane Π has equation $x-3y-2z=1$.

(b) Find the perpendicular distance of Π from the origin. [1]

$$\frac{1}{\sqrt{1^2+(-3)^2+(-2)^2}} = \boxed{\frac{1}{\sqrt{14}}}$$

(c) Find the acute angle between the planes OAB and Π. [3]

$\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} = \sqrt{3}\sqrt{14}\cos\theta$

$\cos\theta = \dfrac{6}{\sqrt{3}\sqrt{14}}$

$\boxed{\theta = 22.2^\circ}$

(d) Find an equation for the common perpendicular to the lines OC and AB. [10]

$\begin{vmatrix} i & j & k \\ 2 & 1 & -7 \\ -2 & -3 & 1 \end{vmatrix} = \begin{bmatrix} -20 \\ 12 \\ -4 \end{bmatrix} \sim \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$

$\overrightarrow{OP} = \lambda \begin{pmatrix} 2 \\ 1 \\ -7 \end{pmatrix} = \begin{pmatrix} 2\lambda \\ \lambda \\ -7\lambda \end{pmatrix}$

$\overrightarrow{OQ} = \begin{pmatrix} 2-2\mu \\ 2-3\mu \\ \mu \end{pmatrix}$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 2-2\mu-2\lambda \\ 2-3\mu-\lambda \\ \mu+7\lambda \end{pmatrix}$

$\begin{pmatrix} 2-2\mu-2\lambda \\ 2-3\mu-\lambda \\ \mu+7\lambda \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix} = 0$

$14\mu + 14\lambda = 10$

$$\begin{pmatrix} 2 - 2\mu - 2\lambda \\ 2 - 3\mu - \lambda \\ \mu + 7\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -7 \end{pmatrix} = 0$$

$14\mu + 54\lambda = 6$

$\lambda = -\frac{1}{10}$

$$\vec{OP} = \begin{pmatrix} -\frac{1}{5} \\ -1/10 \\ 7/10 \end{pmatrix}$$

$$\boxed{r = \begin{pmatrix} -1/5 \\ -1/10 \\ 7/10 \end{pmatrix} + t\begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}}$$

Additional Page

If you use the following lined page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.

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