Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

5202311540

FURTHER MATHEMATICS **9231/13**

Paper 1 Further Pure Mathematics 1 **May/June 2021**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **20** pages. Any blank pages are indicated.

[Turn over

1 **(a)** Show that

$$\tan(r+1) - \tan r = \frac{\sin 1}{\cos(r+1)\cos r}.$$ [2]

$\frac{\sin(r+1)}{\cos(r+1)} - \frac{\sin r}{\cos r} = \frac{\sin(r+1)\cos r - \sin r\cos(r+1)}{\cos r\cos(r+1)} = \frac{\sin(r+1-r)}{\cos(r+1)\cos r}$

$= \frac{\sin 1}{\cos(r+1)\cos r}$ Hence shown.

Let $u_r = \dfrac{1}{\cos(r+1)\cos r}$.

(b) Use the method of differences to find $\sum_{r=1}^{n} u_r$. [3]

$\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} \tan(r+1) - \tan(r)$

$r=1$: $\tan 2 - \tan 1$

$r=2$: $\tan 3 - \tan 2$

$r=n$: $\tan(n+1) - \tan n$

$\Rightarrow \boxed{\dfrac{\tan(n+1) - \tan n}{\sin 1}}$

(c) Explain why the infinite series $u_1 + u_2 + u_3 + \dots$ does not converge. [1]

$\tan(n+1)$ oscillates as $n \to \infty$

so $u_1 + u_2 + u_3 + \dots$ does not converge.

2 The cubic equation $2x^3 - 4x^2 + 3 = 0$ has roots α, β, γ. Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(a) State the value of S_1 and find the value of S_2. [3]

$\boxed{S_1 = 2}$

$S_2 = S_1^2 - 2(0)$

$= 2^2 - 0$

$\boxed{S_2 = 4}$

(b) **(i)** Express S_{n+3} in terms of S_{n+2} and S_n. [1]

$\boxed{S_{n+3} = 2S_{n+2} - \frac{3}{2}S_n}$

(ii) Hence, or otherwise, find the value of S_4. [2]

$S_4 = 2S_3 - \frac{3}{2}S_1 = 2(2S_2 - \frac{3}{2}S_0) - \frac{3}{2}S_1$

$\boxed{S_4 = 4}$

(c) Use the substitution $y = S_1 - x$, where S_1 is the numerical value found in part **(a)**, to find and simplify an equation whose roots are $\alpha+\beta$, $\beta+\gamma$, $\gamma+\alpha$. [3]

$y = 2 - x$

$x = 2 - y$

$2(2-y)^3 - 4(2-y)^2 + 3 = 0$

$\boxed{2y^3 - 8y^2 + 8y - 3 = 0}$

(d) Find the value of $\dfrac{1}{\alpha+\beta} + \dfrac{1}{\beta+\gamma} + \dfrac{1}{\gamma+\alpha}$. [2]

$\dfrac{8/2}{-3/2} = \boxed{\dfrac{8}{3}}$

3 **(a)** Prove by mathematical induction that, for all positive integers n,

$$\sum_{r=1}^{n}\left(5r^4+r^2\right)=\tfrac{1}{2}n^2(n+1)^2(2n+1).$$ [6]

for $n=1$

$\sum_{r=1}^{1} 5r^4+r^2 = 5(1)^4+1^2 = 5+1=6$ (LHS)

$\frac{1}{2}(1)^2(1+1)^2(2+1) = \frac{1}{2}(2)^2(3) = 6$ (RHS)

LHS = RHS so true

Assume true for $n=k$

$\sum_{r=1}^{k}(5r^4+r^2) = \frac{1}{2}k^2(k+1)^2(2k+1)$

Prove for $n=k+1$

$\sum_{r=1}^{k+1}(5r^4+r^2) = \sum_{r=1}^{k}(5r^4+r^2) + 5(k+1)^4+(k+1)^2$

$= \frac{1}{2}(k+1)^2(2k^3+k^2+10(k^2+2k+1)+2)$

$= \frac{1}{2}(k+1)^2(2k^3+11k^2+20k+12)$

$= \frac{1}{2}(k+1)^2(k+2)^2(2k+3)$

Hence induction is complete.

(b) Use the result given in part **(a)** together with the List of formulae (MF19) to find $\sum_{r=1}^{n} r^4$ in terms of n, fully factorising your answer. [3]

$5\sum_{r=1}^{n} r^4 = \frac{1}{2}n^2(n+1)^2(2n+1) - \frac{1}{6}n(n+1)(2n+1)$

$\sum_{r=1}^{n} r^4 = \frac{1}{30}n(n+1)(2n+1)(3n(n+1)-1)$

$\boxed{\sum_{r=1}^{n} r^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)}$

4 The matrices **A**, **B** and **C** are given by

$$\mathbf{A} = \begin{pmatrix} 2 & k & k \\ 5 & -1 & 3 \\ 1 & 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \quad \mathbf{C} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 2 & 0 \end{pmatrix},$$

where k is a real constant.

(a) Find **CAB**. [3]

$$\begin{pmatrix} 0 & 1 & 1 \\ -1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 2 & k & k \\ 5 & -1 & 3 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 2+k & k \\ 8 & -1 \\ 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 10 & -1 \\ 14-k & -k-2 \end{pmatrix}$$

(b) Given that **A** is singular, find the value of k. [3]

$$2\begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix} - k\begin{vmatrix} 5 & 3 \\ 1 & 1 \end{vmatrix} + k\begin{vmatrix} 5 & -1 \\ 1 & 0 \end{vmatrix} = 0$$

$$-2 - 2k + k = 0$$

$$-k = 2$$

$$\boxed{k = -2}$$

(c) Using the value of k from part **(b)**, find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **CAB**. [5]

$$\begin{pmatrix} 10 & -1 \\ 16 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10x-y \\ 16x \end{pmatrix}$$

$10x - mx = 16mx$

$16m = 10m - m^2$

$m^2 - 10m + 16 = 0$

$m^2 - 8m - 2m + 16 = 0$

$m(m-8) - 2(m-8) = 0$

$(m-2)(m-8) = 0$

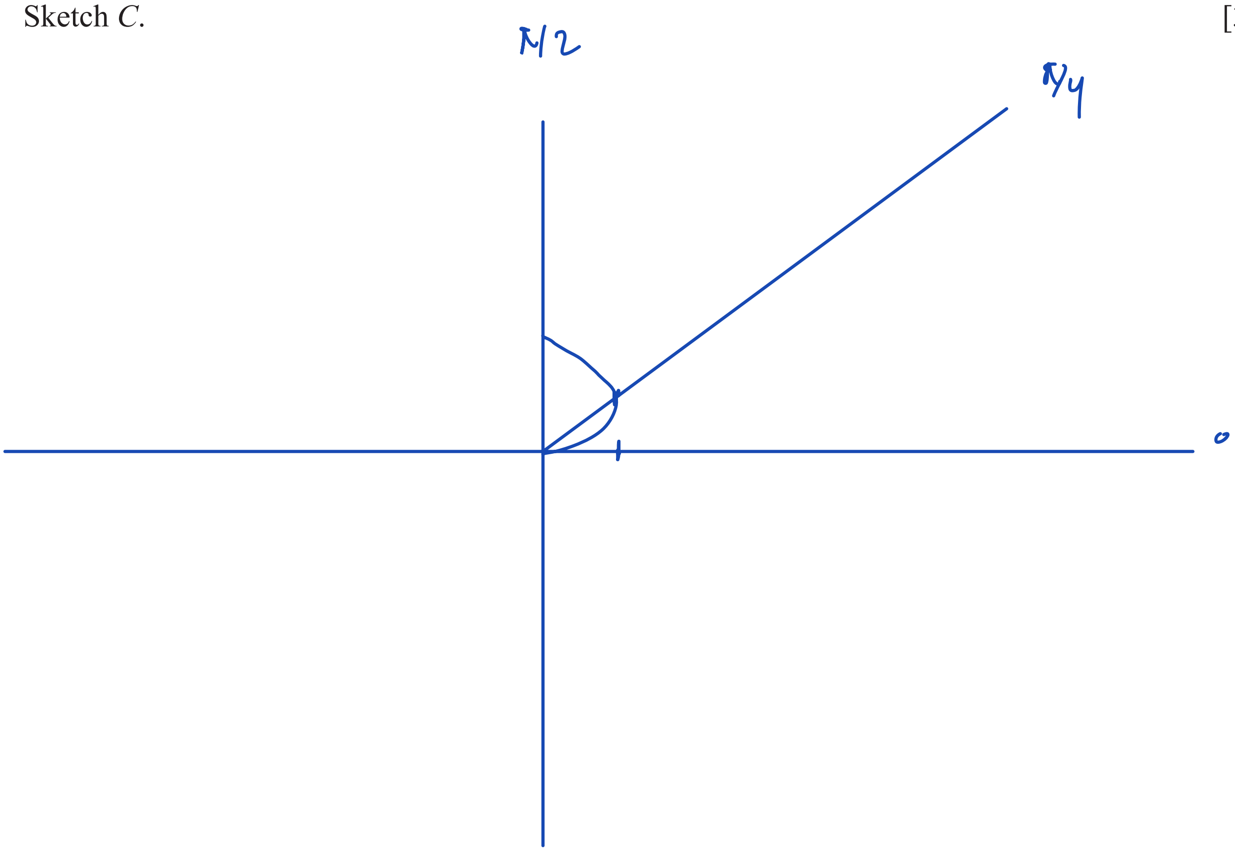
$m-2=0 \qquad m-8=0$

$m=2 \qquad m=8$

$\boxed{y=2x,\ y=8x}$

5 The curve C has polar equation $r = \frac{1}{\pi - \theta} - \frac{1}{\pi}$, where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) Sketch C. [3]



(b) Show that the area of the region bounded by the half-line $\theta = \frac{1}{2}\pi$ and C is $\frac{3 - 4\ln 2}{4\pi}$. [6]

$\frac{1}{2}\int_0^{\pi/2} \left(\frac{1}{\pi-\theta} - \frac{1}{\pi}\right)^2 d\theta$

$\frac{1}{2}\int_0^{\pi/2} \frac{1}{(\pi-\theta)^2} - \frac{2}{\pi(\pi-\theta)} + \frac{1}{\pi^2}\, d\theta$

$\frac{1}{2}\left[\frac{1}{\pi-\theta} + \frac{2}{\pi}\ln \pi-\theta + \frac{\theta}{\pi^2}\right]_0^{\pi/2}$

$\frac{1}{2}\left(\frac{2}{\pi} + \frac{2}{\pi}\ln\frac{\pi}{2} + \frac{1}{2\pi} - \left(\frac{1}{\pi} + \frac{2}{\pi}\ln\pi\right)\right) = \frac{1}{2}\left(\frac{3}{2\pi} + \frac{2}{\pi}\ln 2\right)$

$\boxed{= \frac{3 - 4\ln 2}{4\pi}}$

6 The lines l_1 and l_2 have equations $\mathbf{r} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k} + s(2\mathbf{i} - 3\mathbf{j})$ and $\mathbf{r} = 3\mathbf{i} - 2\mathbf{k} + t(3\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ respectively.

The plane Π_1 contains l_1 and the point P with position vector $-2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$.

(a) Find an equation of Π_1, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$. [2]

$-i - 2j + k - (-2i - 2j + 4k) = i - 3k$

$\boxed{r = -2i - 2j + 4k + \lambda(2i - 3j) + \mu(i - 3k)}$

The plane Π_2 contains l_2 and is parallel to l_1.

(b) Find an equation of Π_2, giving your answer in the form $ax + by + cz = d$. [4]

$$\vec{n} = \begin{vmatrix} i & j & k \\ 2 & -3 & 0 \\ 3 & -1 & 3 \end{vmatrix} = \begin{pmatrix} -9 \\ -6 \\ 7 \end{pmatrix}$$

$-9x - 6y + 7z \Rightarrow -9(3) - 6(0) + 7(-2) = -41$

$-9x - 6y + 7z = -41$

$\boxed{9x + 6y - 7z = 41}$

(c) Find the acute angle between Π_1 and Π_2. [5]

$$\underset{\sim}{n} = \begin{vmatrix} i & j & k \\ 2 & -3 & 0 \\ 1 & 0 & -3 \end{vmatrix} = \begin{pmatrix} 9 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 6 \\ 7 \end{pmatrix} = \sqrt{14}\sqrt{166}\cos\theta$$

$$\cos\theta = \frac{32}{\sqrt{14}\sqrt{166}}$$

$$\boxed{\theta = 48.4^\circ}$$

(d) The point Q is such that $\overrightarrow{OQ} = -5\overrightarrow{OP}$.

Find the position vector of the foot of the perpendicular from the point Q to Π_2. [4]

$\overrightarrow{OF} = \overrightarrow{OQ} + \overrightarrow{QF} = -5\begin{pmatrix} -2 \\ -2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 9 \\ 6 \\ -7 \end{pmatrix} = \begin{pmatrix} 10+9t \\ 10+6t \\ -20-7t \end{pmatrix}$

$9(10+9t) + 6(10+6t) - 7(-20-7t) = 41$

$290 + 166t = 41$

$t = -\frac{3}{2}$

$\boxed{\overrightarrow{OF} = \begin{bmatrix} -7/2 \\ 1 \\ -19/2 \end{bmatrix}}$

7 The curve C has equation $y = \dfrac{x^2 - x - 3}{1 + x - x^2}$.

(a) Find the equations of the asymptotes of C. [2]

$x^2 - x - 1 = 0$

$x = \dfrac{1 \pm \sqrt{5}}{2}$

$\boxed{x = \dfrac{1+\sqrt{5}}{2}, \quad x = \dfrac{1-\sqrt{5}}{2}, \quad y = -1}$

(b) Find the coordinates of any stationary points on C. [3]

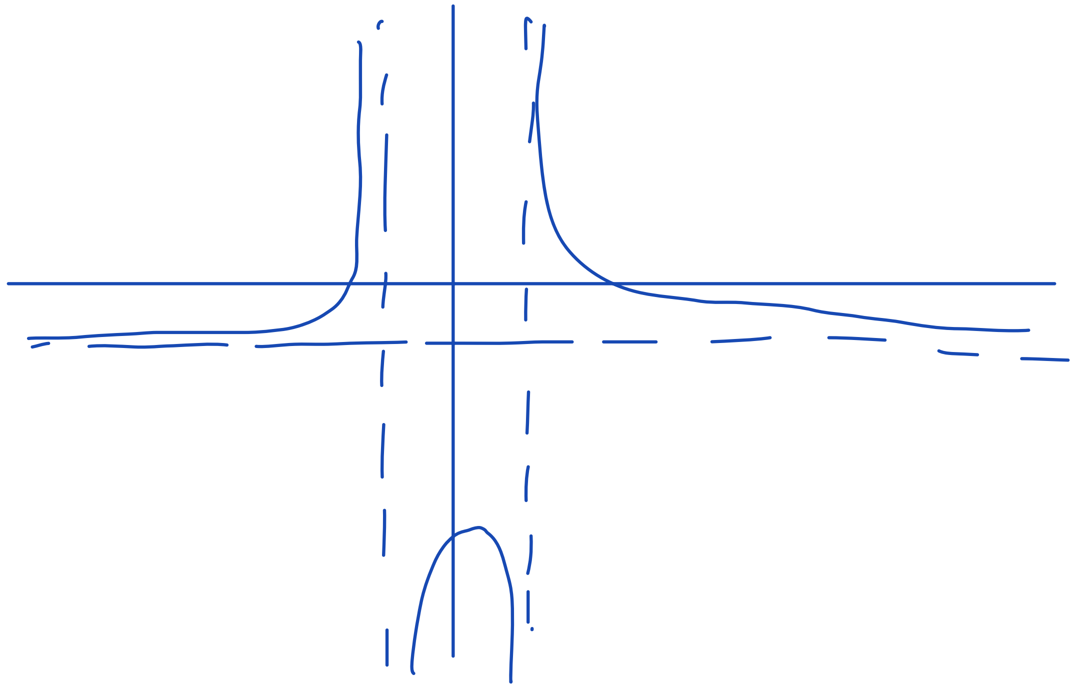
$\dfrac{dy}{dx} = \dfrac{(1+x-x^2)(2x-1) - (x^2-x-3)(1-2x)}{1+x-x^2} = 0$

$(2x-1)(-2) = 0$

$x = 1/2$

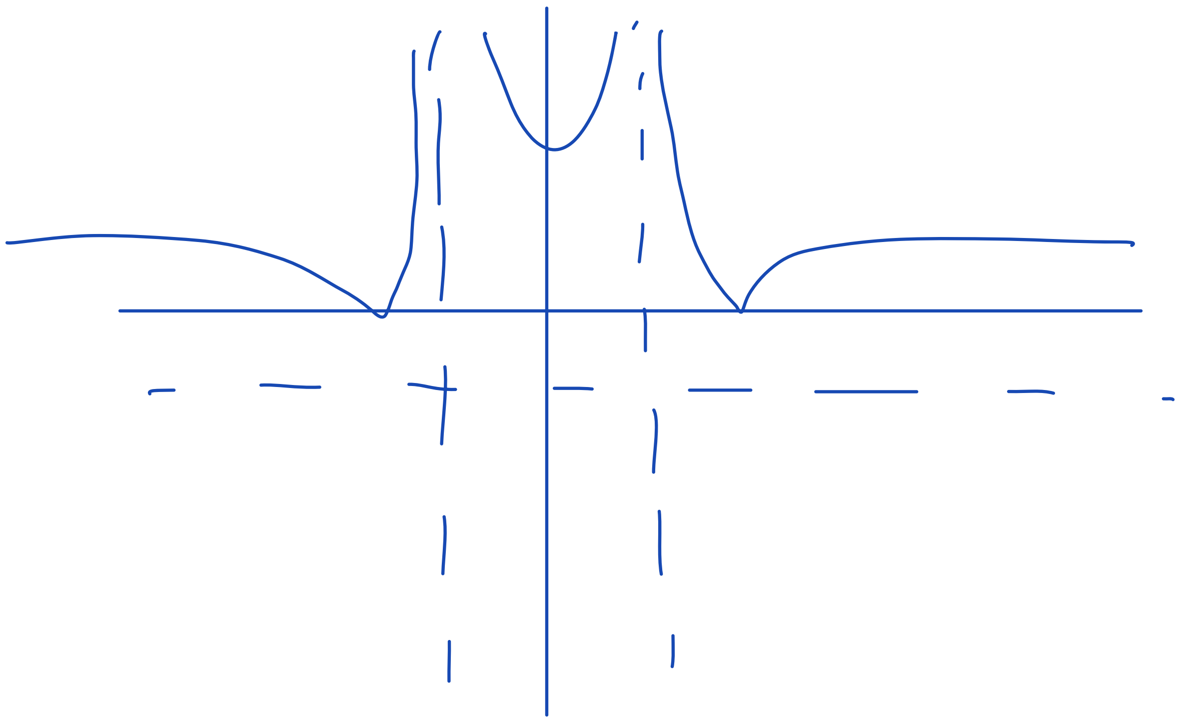
$\boxed{\left(\frac{1}{2}, \frac{-13}{5}\right)}$

(c) Sketch C, stating the coordinates of the intersections with the axes. [3]



$\left(\frac{1}{2}+\frac{1}{2}\sqrt{3}, 0\right)$ $\left(\frac{1}{2}-\frac{1}{2}\sqrt{3}, 0\right)$ $(0, -3)$

(d) Sketch the curve with equation $y = \left|\frac{x^2 - x - 3}{1 + x - x^2}\right|$ and find in exact form the set of values of x for which $\left|\frac{x^2 - x - 3}{1 + x - x^2}\right| < 3$. [6]



$\frac{x^2-x-3}{1+x-x^2}=3$

$2x^2-2x-3=0$

$x=\frac{1}{2}\pm\frac{1}{2}\sqrt{7}$

$\frac{x^2-x-3}{1+x-x^2}=-3$

$-2x^2+2x=0$

$x=0, 1$

$\boxed{x<\frac{1}{2}-\frac{1}{2}\sqrt{7}, \quad 0<x<1, \quad x>\frac{1}{2}+\frac{1}{2}\sqrt{7}}$

Additional Page

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