Cambridge Assessment
International Education

Cambridge International AS & A Level

CANDIDATE NAME		
CENTRE NUMBER		CANDIDATE NUMBER

9936421715

FURTHER MATHEMATICS **9231/11**

Paper 1 Further Pure Mathematics 1 **May/June 2021**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Any blank pages are indicated.

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1 Prove by mathematical induction that $2^{4n} + 31^n - 2$ is divisible by 15 for all positive integers n. [6]

for $n=1$

$2^4 + 31 - 2 = 16 + 2 - 31 = 45 = 15(3)$

so true.

Assume true for $n=k$

$2^{4k} + 31^k - 2$

Prove for $n=k+1$

$2^{4k+4} + 31^{k+1} - 2$

$16 \cdot 2^{4k} + 31 \cdot 31^k - 2$

$(15+1) \cdot 2^{4k} + (30+1) \cdot 31^k - 2$

$15 \cdot 2^{4k} + 30 \cdot 31^k + 2^{4k} + 31^k - 2$

$15 \cdot 2^{4k} + 30 \cdot 31^k + f_1(k)$

$15\left[2^{4k} + 2 \cdot 3^k + f_2(k)\right]$

Hence induction is complete.

2 **(a)** Use standard results from the List of formulae (MF19) to find $\sum_{r=1}^{n}\left(1-r-r^2\right)$ in terms of n, simplifying your answer. [3]

$\sum_{r=1}^{n} 1 - \sum_{r=1}^{n} r - \sum_{r=1}^{n} r^2$

$n - \frac{1}{2}n(n+1) - \frac{1}{6}n(n+1)(2n+1)$

$\boxed{\frac{1}{3}n - n^2 - \frac{1}{3}n^3}$

(b) Show that

$$\frac{1-r-r^2}{(r^2+2r+2)(r^2+1)} = \frac{r+1}{(r+1)^2+1} - \frac{r}{r^2+1}$$

and hence use the method of differences to find $\sum_{r=1}^{n} \frac{1-r-r^2}{(r^2+2r+2)(r^2+1)}$. [5]

$$\frac{(r+1)(r^2+1) - r(r^2+2r+2)}{(r^2+2r+2)(r^2+1)}$$

$$\frac{\cancel{r^3}+r+r^2+1-\cancel{r^3}-2r^2-2r}{((r+1)^2+1)(r^2+1)} = \frac{1-r-r^2}{(r^2+2r+2)(r^2+1)}$$

$$\sum_{r=1}^{n}\left(\frac{r+1}{(r+1)^2+1} - \frac{r}{r^2+1}\right) = \frac{2}{5} - \frac{1}{2} + \frac{3}{10} - \frac{2}{5} + \frac{4}{17} - \frac{3}{10}$$

$$+ \frac{n+1}{(n+1)^2+1} - \frac{n}{n^2+1}$$

$$= \boxed{-\frac{1}{2} + \frac{n+1}{(n+1)^2+1}}$$

(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1-r-r^2}{(r^2+2r+2)(r^2+1)}$. [1]

$-1/2$.

3 The equation $x^4 - 2x^3 - 1 = 0$ has roots α, β, γ, δ.

(a) Find a quartic equation whose roots are α^3, β^3, γ^3, δ^3 and state the value of $\alpha^3 + \beta^3 + \gamma^3 + \delta^3$. [4]

$y = x^3$

$x = y^{1/3}$

$y^{4/3} - 2y - 1 = 0$

$y^4 = (2y+1)^3$

$y^4 = 8y^3 + 12y^2 + 6y + 1$

$\boxed{y^4 - 8y^3 - 12y^2 - 6y - 1 = 0}$

$\boxed{\alpha^3 + \beta^3 + \gamma^3 + \delta^3 = 8}$

(b) Find the value of $\frac{1}{\alpha^3}+\frac{1}{\beta^3}+\frac{1}{\gamma^3}+\frac{1}{\delta^3}$. [3]

$$\frac{1}{\alpha^3}+\frac{1}{\beta^3}+\frac{1}{\gamma^3}+\frac{1}{\delta^3} = \frac{\alpha^3\beta^3\gamma^3+\alpha^3\beta^3\delta^3+\beta^3\gamma^3\delta^3+\alpha^3\gamma^3\delta^3}{\alpha^3\beta^3\gamma^3\delta^3} = \frac{6}{-1}$$

$$\boxed{\frac{1}{\alpha^3}+\frac{1}{\beta^3}+\frac{1}{\gamma^3}+\frac{1}{\delta^3} = -6}$$

(c) Find the value of $\alpha^4+\beta^4+\gamma^4+\delta^4$. [2]

$$\alpha^4+\beta^4+\gamma^4+\delta^4 = 2(\alpha^3+\beta^3+\gamma^3+\delta^3)+4$$

$$\boxed{\alpha^4+\beta^4+\gamma^4+\delta^4 = 20}$$

4 The matrix **M** represents the sequence of two transformations in the x-y plane given by a rotation of 60° anticlockwise about the origin followed by a one-way stretch in the x-direction, scale factor d ($d \neq 0$).

(a) Find **M** in terms of d. [4]

$$\begin{pmatrix} \cos 60 & -\sin 60 \\ \sin 60 & \cos 60 \end{pmatrix} = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$$

$$\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{M = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} = \begin{pmatrix} d/2 & -d\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}}$$

(b) The unit square in the x-y plane is transformed by **M** onto a parallelogram of area $\frac{1}{2}d^2$ units^2.

Show that $d = 2$. [2]

$$d = \frac{1}{2}d^2$$
$$2d = d^2$$
$$d^2 - 2d = 0$$
$$d(d-2) = 0$$
$$d \neq 0 \Rightarrow \boxed{d = 2}$$

The matrix $\mathbf{N}$ is such that $\mathbf{MN} = \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

(c) Find $\mathbf{N}$. [3]

$M^{-1} = \frac{1}{2}\begin{bmatrix} 1/2 & \sqrt{3} \\ -\sqrt{3}/2 & 1 \end{bmatrix}$

$N = M^{-1}\begin{pmatrix} 1 & 1 \\ 1/2 & 1/2 \end{pmatrix} = \frac{1}{2}\begin{bmatrix} 1/2 & \sqrt{3} \\ -\sqrt{3}/2 & 1 \end{bmatrix}\begin{pmatrix} 1 & 1 \\ 1/2 & 1/2 \end{pmatrix}$

$\boxed{N = \frac{1}{4}\begin{pmatrix} 1+\sqrt{3} & 1+\sqrt{3} \\ 1-\sqrt{3} & 1-\sqrt{3} \end{pmatrix}}$

(d) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by $\mathbf{MN}$. [5]

$\begin{pmatrix} 1 & 1 \\ 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ \frac{1}{2}x + \frac{1}{2}y \end{pmatrix}$

$\frac{1}{2}x + \frac{1}{2}mx = m(x+mx)$

$\frac{1}{2} + \frac{1}{2}m = m + m^2$

$m^2 + \frac{1}{2}m - \frac{1}{2} = 0$

$2m^2 + m - 1 = 0$

$2m^2 + 2m - m - 1 = 0$

$2m(m+1) - 1(m+1) = 0$

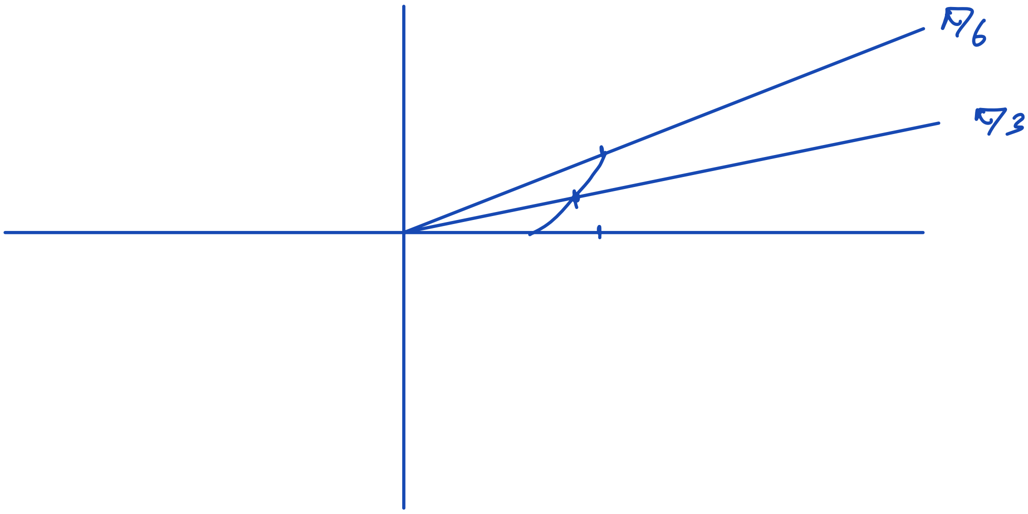
$m = \frac{1}{2}, -1$

$\boxed{y = \frac{1}{2}x, \quad y = -x}$

5 The curve C has polar equation $r = a\cot\left(\frac{1}{3}\pi - \theta\right)$, where a is a positive constant and $0 \leqslant \theta \leqslant \frac{1}{6}\pi$.

It is given that the greatest distance of a point on C from the pole is $2\sqrt{3}$.

(a) Sketch C and show that $a = 2$. [3]



$a\cot\frac{\pi}{6} = 2\sqrt{3} \Rightarrow \boxed{a=2}$

(b) Find the exact value of the area of the region bounded by C, the initial line and the half-line $\theta = \frac{1}{6}\pi$. [4]

$A = \frac{1}{2}\int_0^{\pi/6} 4\cot^2\left(\frac{1}{3}\pi - \theta\right)\, d\theta$

$2\int_0^{\pi/6} \operatorname{cosec}^2\left(\frac{\pi}{3} - \theta\right) - 1\, d\theta$

$= 2\left[\cot\left(\frac{\pi}{3} - \theta\right) - \theta\right]_0^{\pi/6}$

$= 2\left(\sqrt{3} - \frac{\pi}{6} - \frac{1}{3}\sqrt{3}\right) = \boxed{\frac{4}{3}\sqrt{3} - \frac{\pi}{3}}$

(c) Show that C has Cartesian equation $2\left(x+y\sqrt{3}\right)=\left(x\sqrt{3}-y\right)\sqrt{x^2+y^2}$. [3]

$$r = \frac{2\left(\cos\frac{\pi}{3}\cos\theta + \sin\frac{\pi}{3}\sin\theta\right)}{\sin\frac{\pi}{3}\cos\theta - \cos\frac{\pi}{3}\sin\theta}$$

$$\sqrt{x^2+y^2} = \frac{2\left(r\cos\theta + \sqrt{3}r\sin\theta\right)}{\sqrt{3}r\cos\theta - r\sin\theta}$$

$$\boxed{2(x+y\sqrt{3}) = (x\sqrt{3}-y)(\sqrt{x^2+y^2})}$$

6 Let t be a positive constant.

The line l_1 passes through the point with position vector $t\mathbf{i}+\mathbf{j}$ and is parallel to the vector $-2\mathbf{i}-\mathbf{j}$. The line l_2 passes through the point with position vector $\mathbf{j}+t\mathbf{k}$ and is parallel to the vector $-2\mathbf{j}+\mathbf{k}$.

It is given that the shortest distance between the lines l_1 and l_2 is $\sqrt{21}$.

(a) Find the value of t. [5]

$$\begin{pmatrix}0\\1\\t\end{pmatrix}-\begin{pmatrix}t\\1\\0\end{pmatrix}=t\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

$$\vec{n}=\begin{vmatrix}i & j & k\\ -2 & -1 & 0\\ 0 & -2 & 1\end{vmatrix}=\begin{pmatrix}-1\\2\\4\end{pmatrix}$$

$$\frac{t}{\sqrt{1^2+2^2+4^2}}\left|\begin{pmatrix}-1\\0\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\4\end{pmatrix}\right|=\sqrt{21}\Rightarrow \boxed{t=4.2}$$

The plane Π_1 contains l_1 and is parallel to l_2.

(b) Write down an equation of Π_1, giving your answer in the form $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}+\mu\mathbf{c}$. [1]

$$\boxed{r=\frac{21}{5}i+j+\lambda(2i+j)+\mu(-2j+k)}$$

The plane Π_2 has Cartesian equation $5x - 6y + 7z = 0$.

(c) Find the acute angle between l_2 and Π_2. [3]

$$\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -6 \\ 7 \end{pmatrix} = \sqrt{5}\sqrt{110}\cos\theta$$

$$\cos\theta = \frac{19}{\sqrt{5}\sqrt{110}}$$

$$\boxed{\theta = 90 - \theta \Rightarrow 54.1^\circ}$$

(d) Find the acute angle between Π_1 and Π_2. [3]

$$\begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -6 \\ 7 \end{pmatrix} = \sqrt{21}\sqrt{110}\cos\theta$$

$$\cos\theta = \frac{11}{\sqrt{21}\sqrt{110}}$$

$$\boxed{\Rightarrow 76.8^\circ}$$

7 The curve C has equation $y = \dfrac{x^2 + x + 9}{x + 1}$.

(a) Find the equations of the asymptotes of C. [3]

$\boxed{x = -1}$

$y = \dfrac{x(x+1) + 9}{x+1} = x + \dfrac{9}{x+1}$

$\boxed{y = x}$

(b) Find the coordinates of the stationary points on C. [4]

$\dfrac{dy}{dx} = 1 - \dfrac{9}{(x+1)^2}$

$\Rightarrow \quad (x+1)^2 = 9$

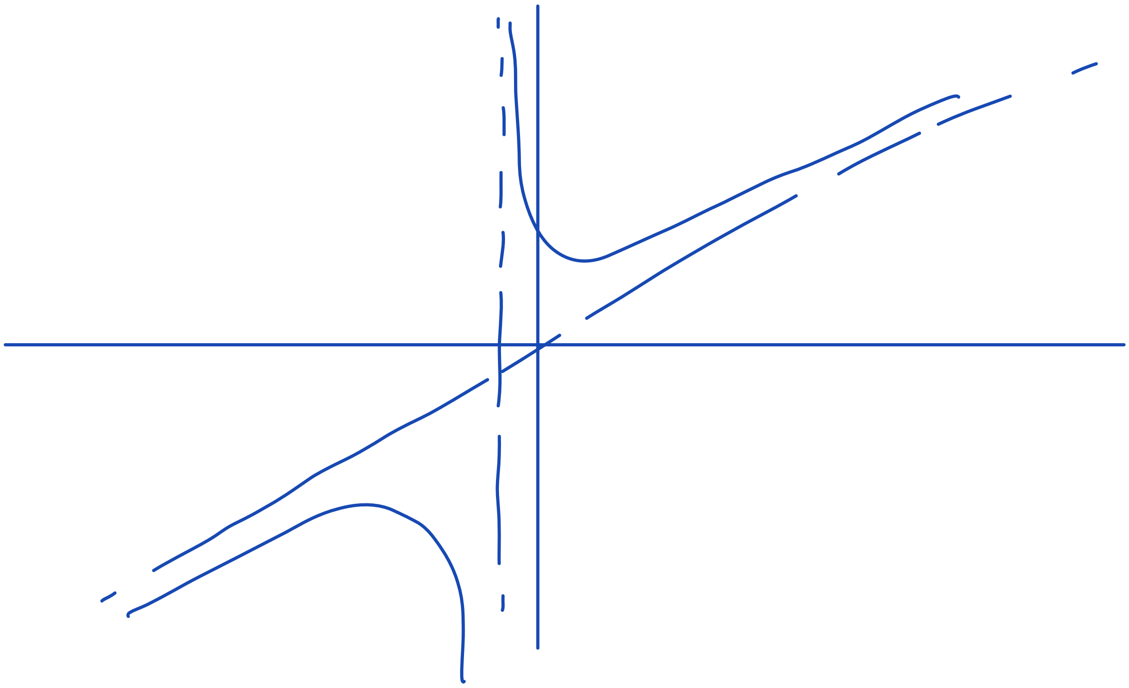
$x + 1 = \pm 3$

$x + 1 = 3$ $\qquad\qquad$ $x + 1 = -3$

$x = 2$ $\qquad\qquad$ $x = -4$

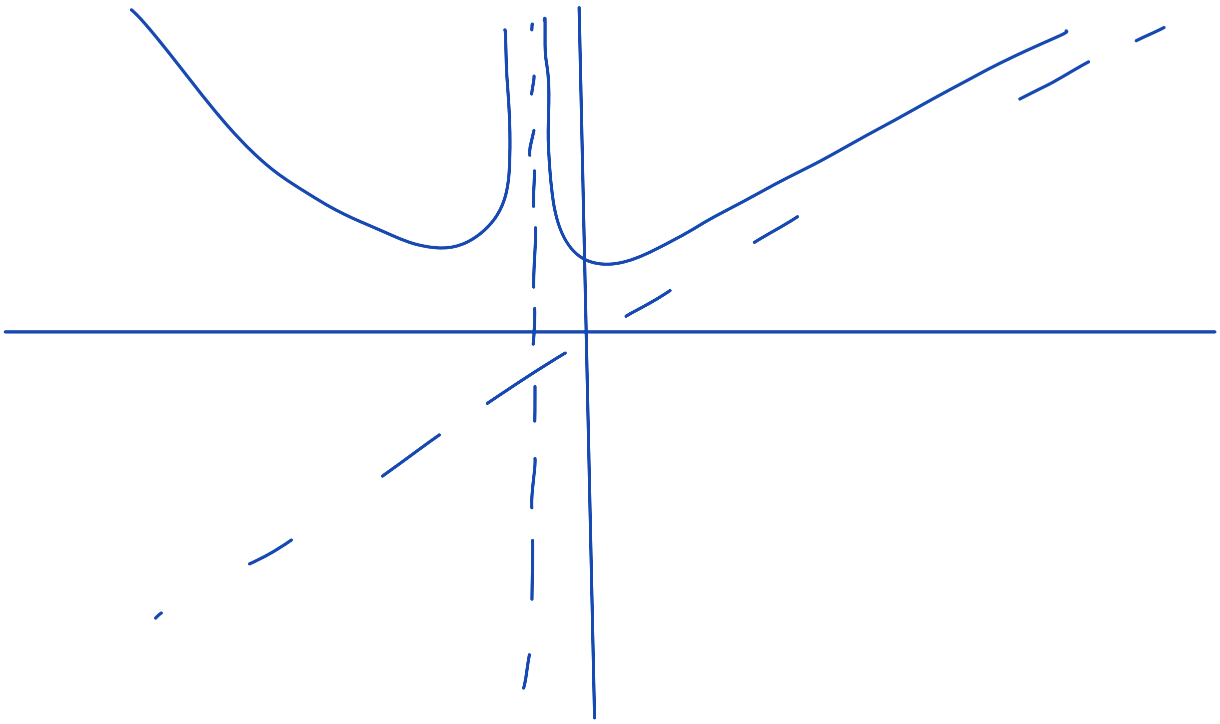
$\boxed{(2, 5) \qquad (-4, -7)}$

(c) Sketch C, stating the coordinates of any intersections with the axes. [3]



(0,9)

(d) Sketch the curve with equation $y = \left|\dfrac{x^2+x+9}{x+1}\right|$ and find the set of values of x for which $2|x^2+x+9| > 13|x+1|$. [5]



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Additional Page

If you use the following lined page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.

Q7(d)

$x^2+x+9=\frac{13}{2}(x+)$ $\quad$ $x^2+x+9=-\frac{13}{2}(x+1)$

$x^2-\frac{11}{2}x+\frac{5}{2}=0$ $\quad$ $x^2+\frac{15}{2}x+\frac{31}{2}=0$

$x=\frac{1}{2},5$

$\boxed{x<\tfrac{1}{2} \qquad x>5}$