Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

6146662382

FURTHER MATHEMATICS **9231/12**

Paper 1 Further Pure Mathematics 1 **October/November 2022**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **20** pages. Any blank pages are indicated.

[Turn over

1 **(a)** Use the list of formulae (MF19) to find $\sum_{r=1}^{n} r(r+2)$ in terms of n, simplifying your answer. [2]

$\sum_{r=1}^{n} r^2+2r = \sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r$

$= \frac{1}{6}n(n+1)(2n+1) + n(n+1)$

$= \frac{1}{6}n(n+1)\left[2n+1 + 6\right]$

$\boxed{\sum_{r=1}^{n} r(r+2) = \frac{1}{6}n(n+1)(2n+7)}$

(b) Express $\frac{1}{r(r+2)}$ in partial fractions and hence find $\sum_{r=1}^{n} \frac{1}{r(r+2)}$ in terms of n. [4]

$\frac{1}{r(r+2)} = \frac{A}{r} + \frac{B}{r+2}$

$A(r+2) + B(r) = 1$

for $r=-2$: $-2B = 1$, $B = -\frac{1}{2}$

for $r=0$: $2A = 1$, $A = \frac{1}{2}$

$\frac{1}{r(r+2)} = \frac{1}{2r} - \frac{1}{2(r+2)}$

$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{1}{2}\sum_{r=1}^{n} \frac{1}{r} - \frac{1}{r+2}$

$r=1: \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right)$

$r=2: \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right)$

$r=3: \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right)$

$r=4: \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right)$

$r=n-1: \left(\cancel{\frac{1}{n-1}} - \frac{1}{n+1}\right)$

$r=n: \left(\cancel{\frac{1}{n}} - \frac{1}{n+2}\right)$

$\boxed{= \frac{1}{2}\left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)}$

(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{r(r+2)}$. [1]

$\sum_{r=1}^{\infty} \frac{1}{r(r+2)} = \lim_{n \to \infty} \left(\frac{1}{2} \left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) \right)$

$\frac{1}{n+1} \to 0$, $\frac{1}{n+2} \to 0$ As $n \to \infty$

$\therefore \frac{1}{2}\left(\frac{3}{2}\right) = \boxed{\frac{3}{4}}$

$a \quad c \quad d \quad e$

2 The equation $x^4 + 3x^2 + 2x + 6 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}, \frac{1}{\delta^2}$ and state the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}$. [4]

$y = \frac{1}{x^2} \Rightarrow x^2 = \frac{1}{y} \Rightarrow x = y^{-1/2}$

$(y^{-1/2})^4 + 3(y^{-1/2})^2 + 2y^{-1/2} + 6 = 0$

$y^{-2} + 3y^{-1} + 2y^{-1/2} + 6 = 0$

$\frac{1}{y^2} + \frac{3}{y} + \frac{2}{y^{1/2}} + 6 = 0$

$1 + 3y + 2y^{3/2} + 6y^2 = 0$

$[6y^2 + (3y+1)]^2 = [-2y^{3/2}]^2$

$(6y^2)^2 + 2(6y^2)(3y+1) + (3y+1)^2 = 4y^3$

$36y^4 + 12y^2(3y+1) + 9y^2 + 6y + 1 = 4y^3$

$36y^4 + 36y^3 + 12y^2 + 9y^2 + 6y + 1 - 4y^3 = 0$

$\boxed{36y^4 + 32y^3 + 21y^2 + 6y + 1 = 0}$

$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2} = \Sigma\alpha = -\frac{b}{a} = \frac{-32}{36} = -\frac{8}{9}$

$\boxed{\Sigma\alpha = -\frac{8}{9}}$

(b) Find the value of $\beta^2\gamma^2\delta^2 + \alpha^2\gamma^2\delta^2 + \alpha^2\beta^2\delta^2 + \alpha^2\beta^2\gamma^2$. [3]

$\alpha^2\beta^2\gamma^2\delta^2\left[\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{1}{\delta^2}\right]$

$\alpha^2\beta^2\gamma^2\delta^2 = (\Sigma\alpha\beta\gamma\delta)^2 = 6^2 = 36$

$\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{1}{\delta^2} = -\frac{8}{9}$

$\boxed{\beta^2\gamma^2\delta^2 + \alpha^2\gamma^2\delta^2 + \alpha^2\beta^2\delta^2 + \alpha^2\beta^2\gamma^2 = -\frac{8}{9}(36) = -32}$

(c) Find the value of $\frac{1}{\alpha^4}+\frac{1}{\beta^4}+\frac{1}{\gamma^4}+\frac{1}{\delta^4}$. [2]

$\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$

$= \left(-\frac{8}{9}\right)^2 - 2\left(\frac{21}{36}\right)$

$\boxed{\Sigma\alpha^2 = \frac{-61}{162}}$

3 The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$, where k is a constant and $k \neq 0$ or 1.

(a) The matrix $\mathbf{M}$ represents a sequence of two geometrical transformations.

State the type of each transformation, and make clear the order in which they are applied. [2]

Shear in y-direction followed by a stretch parallel to y-axis scale factor k.

(b) Write $\mathbf{M}^{-1}$ as the product of two matrices, neither of which is $\mathbf{I}$. [2]

$M^{-1} = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}^{-1}$

$M^{-1} = \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} \cdot \frac{1}{k}\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$

$\boxed{M^{-1} = \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1/k \end{pmatrix}}$

(c) Show that the invariant points of the transformation represented by $\mathbf{M}$ lie on the line $y = \frac{k^2}{1-k}x$. [4]

$m = \frac{k^2}{1-k}$

$M = \begin{pmatrix} 1 & 0 \\ k^2 & k \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ k^2 & k \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$

$x = X$

$k^2x + kmx = mx$

$k^2 + km = m$

$m - km = k^2$

$m(1-k) = k^2$

$\boxed{m = \frac{k^2}{1-k}}$

(d) The triangle ABC in the x-y plane is transformed by $\mathbf{M}$ onto triangle DEF.

Find the value of k for which the area of triangle DEF is equal to the area of triangle ABC. [2]

$1 = |k|$

$\boxed{k = -1}$

4 The function f is such that $f''(x) = f(x)$.

Prove by mathematical induction that, for every positive integer n,

$$\frac{d^{2n-1}}{dx^{2n-1}}\big(x f(x)\big) = x f'(x) + (2n-1) f(x).$$ [7]

for n=1:

$\frac{d}{dx}(xf(x)) = f(x) + xf'(x)$ (LHS)

$xf'(x) + (2(1)-1)f(x) = f(x) + xf'(x)$ (RHS)

LHS = RHS

Suppose true for n=k:

$\frac{d^{2k-1}}{dx^{2k-1}}(xf(x)) = xf'(x) + (2k-1)f(x)$

Prove true for n=k+1

$\frac{d^{2k+1}}{dx^{2k+1}}(xf(x)) = \frac{d^2}{dx^2}\left(\frac{d^{2k-1}}{dx^{2k-1}}(xf(x))\right)$

$\frac{d}{dx}\left(xf'(x) + (2k-1)f(x)\right)$

$= f'(x) + xf''(x) + (2k-1)f'(x)$

$= \cancel{f'(x)} + xf(x) + 2kf'(x) - \cancel{f'(x)}$

$= xf(x) + 2kf'(x)$

$\frac{d}{dx}(xf(x) + 2kf'(x)) = f(x) + xf'(x) + 2kf''(x)$

$= f(x) + xf'(x) + 2kf(x)$

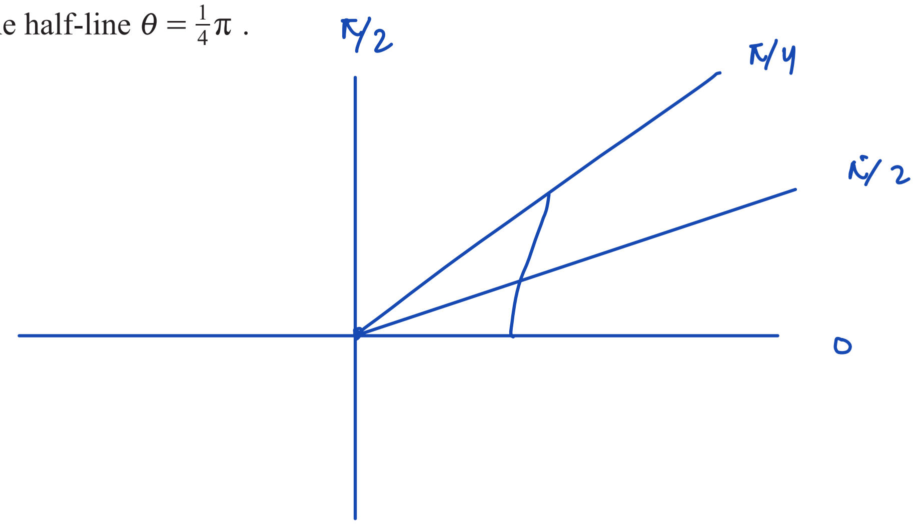
$= xf'(x) + (2k+1)f(x)$

$= xf'(x) + (2(k+1)-1)f(x)$

Hence induction is complete.

5 The curve C has polar equation $r = a\sec^2\theta$, where a is a positive constant and $0 \leqslant \theta \leqslant \frac{1}{4}\pi$.

(a) Sketch C, stating the polar coordinates of the point of intersection of C with the initial line and also with the half-line $\theta = \frac{1}{4}\pi$. [3]



$(a, 0)$ $\left(2a, \frac{\pi}{4}\right)$

(b) Find the maximum distance of a point of C from the initial line. [2]

$y = (2a)\sin\left(\frac{\pi}{4}\right)$ $y = r\sin\theta$

$\boxed{y = a\sqrt{2}}$

(c) Find the area of the region enclosed by C, the initial line and the half-line $\theta = \frac{1}{4}\pi$. [4]

$\frac{1}{2}a^2\int_0^{\pi/4} \sec^4\theta \, d\theta$

$= \frac{1}{2}a^2\int_0^{\pi/4} \sec^2\theta(1 + \tan^2\theta)\, d\theta$

$= \frac{1}{2}a^2\left[\frac{1}{3}\tan^3\theta + \tan\theta\right]_0^{\pi/4}$

$\boxed{= \frac{2}{3}a^2}$

(d) Find, in the form $y = f(x)$, the Cartesian equation of C. [3]

$\sqrt{x^2+y^2} = r$, $x = r\cos\theta$, $y = r\sin\theta$

$r^2\cos^2\theta = ar$

$x^2 = a\sqrt{x^2+y^2}$

$x^4 = a^2(x^2+y^2)$

$$\boxed{y = \sqrt{x^4a^{-2} - x^2}}$$

6 The lines l_1 and l_2 have equations $\mathbf{r} = 2\mathbf{i} + \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = 2\mathbf{j} + 6\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ respectively.

The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2.

(a) Find the length PQ. [5]

$$\begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 5 \end{bmatrix}$$

$$\hat{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ 1 & -1 & 2 \end{vmatrix} = \begin{pmatrix} 2 \\ -4 \\ -3 \end{pmatrix}$$

$$\frac{1}{\sqrt{2^2+4^2+3^2}} \left| \begin{pmatrix} -2 \\ 2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -4 \\ -3 \end{pmatrix} \right| = \boxed{\frac{27}{\sqrt{29}} = 5.01}$$

The plane Π_1 contains PQ and l_1.

The plane Π_2 contains PQ and l_2.

(b) **(i)** Write down an equation of Π_1, giving your answer in the form $\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$. [1]

$r = 2i + k + s(i - j + 2k) + t(2i - 4j - 3k)$

(ii) Find an equation of Π_2, giving your answer in the form $ax + by + cz = d$. [4]

$$\underset{\sim}{n} = \begin{vmatrix} i & j & k \\ 1 & 2 & -2 \\ 2 & -4 & -3 \end{vmatrix} = \begin{pmatrix} -14 \\ -1 \\ -8 \end{pmatrix} \approx \begin{pmatrix} 14 \\ 1 \\ 8 \end{pmatrix}$$

$-14x - y - 8z = d$

$14(0) + 2 - 8(6) = 50$

$\boxed{14x + y + 8z = 50}$

(c) Find the acute angle between Π_1 and Π_2. [5]

$$\underset{\sim}{n} = \begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 2 & -4 & -3 \end{vmatrix} = \begin{pmatrix} 11 \\ 7 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 11 \\ 7 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 14 \\ 1 \\ 8 \end{pmatrix} = \sqrt{174}\sqrt{261}\cos\theta$$

$$\cos\theta = \frac{145}{\sqrt{174}\sqrt{261}}$$

$\boxed{\theta = 47.1^\circ}$

7 The curve C has equation $y = \dfrac{x^2 - x}{x+1}$.

(a) Find the equations of the asymptotes of C. [3]

$\boxed{x = -1}$

$y = \dfrac{(x+1)(x-2) + 2}{x+1}$

$\boxed{y = x - 2}$

(b) Find the exact coordinates of the stationary points on C. [4]

$\dfrac{dy}{dx} = 1 - \dfrac{2}{(x+1)^2} = 0$

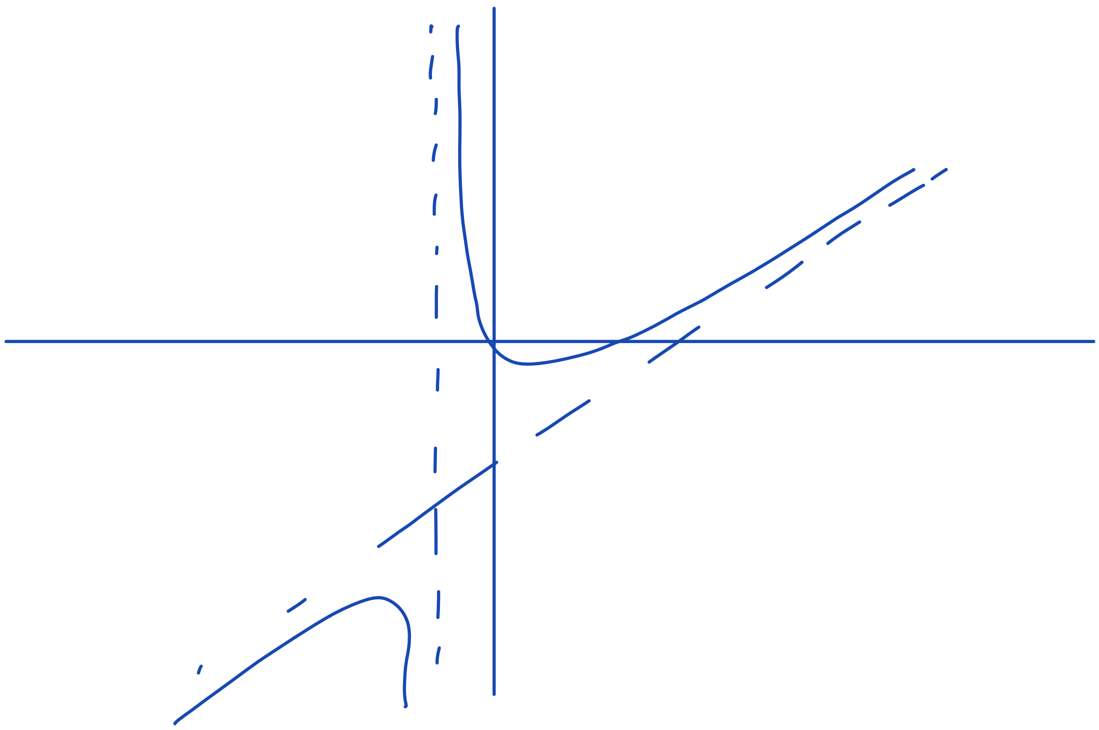
$(x+1)^2 = 2$

$x+1 = \sqrt{2}$ $\quad$ $x+1 = -\sqrt{2}$

$x = \sqrt{2} - 1$ $\quad$ $x = -1 - \sqrt{2}$

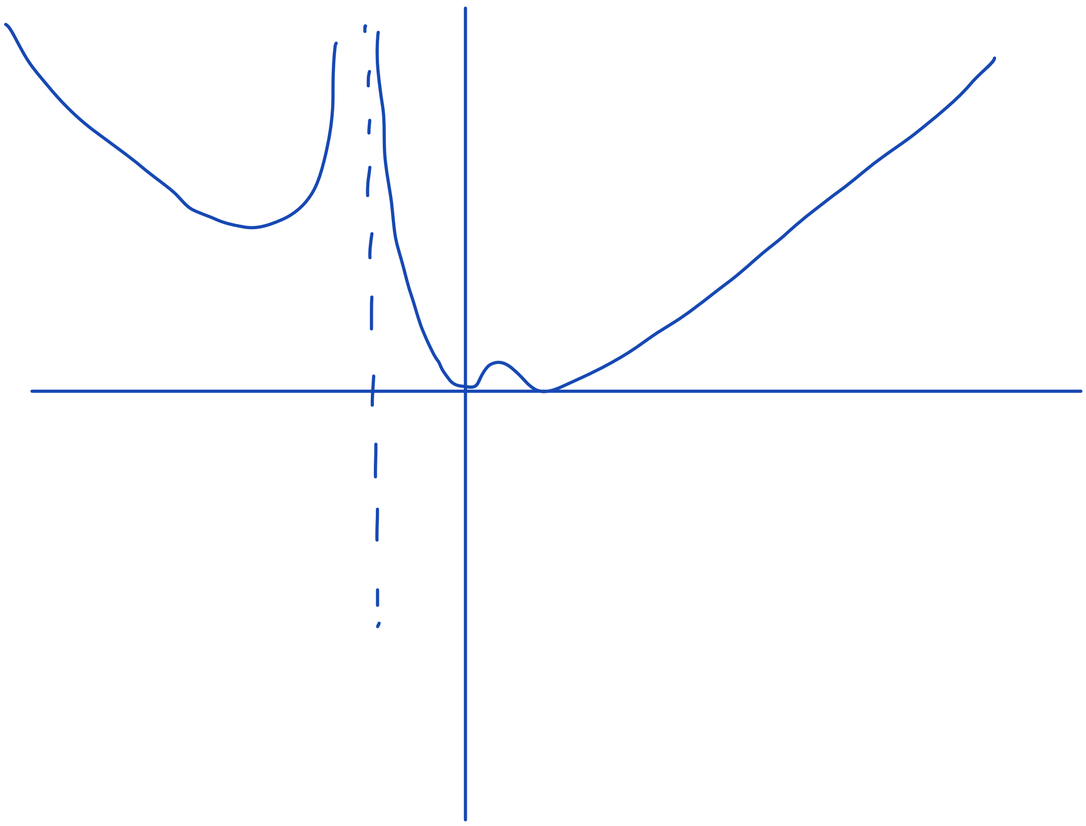
$\boxed{(-1+\sqrt{2},\ -3+2\sqrt{2}) \quad (-1-\sqrt{2},\ -3-2\sqrt{2})}$

(c) Sketch C, stating the coordinates of any intersections with the axes. [3]



(0, 0) (1, 0)

(d) Sketch the curve with equation $y = \left|\frac{x^2 - x}{x+1}\right|$ and find in exact form the set of values of x for which $\left|\frac{x^2 - x}{x+1}\right| < 6$. [5]



$\frac{x^2-x}{x+1}=6$ $\frac{x^2-x}{x+1}=-6$

$x^2-7x-6=0$ $x^2+5x+6=0$

$x=\frac{7}{2}-\frac{1}{2}\sqrt{3}, \frac{7}{2}+\frac{1}{2}\sqrt{3}$ $x=-3, -2$

$\boxed{-3<x<-2 \quad , \quad \frac{7}{2}-\frac{1}{2}\sqrt{3}<x<\frac{7}{2}+\frac{1}{2}\sqrt{3}}$

Additional page

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