Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

6735932410

FURTHER MATHEMATICS **9231/12**

Paper 1 Further Pure Mathematics 1 **May/June 2022**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **20** pages. Any blank pages are indicated.

BLANK PAGE

1 Let a be a positive constant.

(a) Use the method of differences to find $\sum_{r=1}^{n} \frac{1}{(ar+1)(ar+a+1)}$ in terms of n and a. [4]

$\frac{1}{(ar+1)(ar+a+1)} = \frac{A}{ar+1} + \frac{B}{ar+a+1}$ $\quad A(ar+a+1) + B(ar+1) = 1$

$ar+a+1=0 \quad ar+1=0$

Put in: $\quad r = \frac{-a-1}{a} \quad r = -\frac{1}{a}$

$B\left(\cancel{a}\cdot\frac{(-a-1)}{\cancel{a}}+1\right)=1$

$A\left(a\cdot\frac{-1}{a}+a+1\right)=1$

$B(-a\cancel{-1}\cancel{+1})=1 \quad A(\cancel{-1}+a\cancel{+1})=1$

$\boxed{B=-\frac{1}{a}} \quad \boxed{A=\frac{1}{a}} \quad \Rightarrow \frac{1}{a}\left[\frac{1}{ar+1}-\frac{1}{ar+a+1}\right]$

$\sum_{r=1}^{n}\frac{1}{(ar+1)(ar+a+1)} = \frac{1}{a}\sum_{r=1}^{n}\frac{1}{ar+1}-\frac{1}{ar+a+1}$

$r=1: \left(\frac{1}{a+1}-\cancel{\frac{1}{2a+1}}\right) \quad \Rightarrow \frac{1}{a}\left[\frac{1}{a+1}-\frac{1}{a(n+1)+1}\right]$

$r=2: \left(\cancel{\frac{1}{2a+1}}-\cancel{\frac{1}{3a+1}}\right)$

$r=3: \left(\cancel{\frac{1}{3a+1}}-\cancel{\frac{1}{4a+1}}\right)$

$r=n-1: \left(\cancel{\frac{1}{a(n-1)+1}}-\cancel{\frac{1}{na+1}}\right)$

$r=n: \left(\cancel{\frac{1}{na+1}}-\frac{1}{na+a+1}\right)$

(b) Find the value of a for which $\sum_{r=1}^{\infty} \frac{1}{(ar+1)(ar+a+1)} = \frac{1}{6}$. [3]

$\frac{1}{a^2+a}=\frac{1}{6}$

$a^2+a-6=0$

$a^2+3a-2a-6=0$

$(a+3)(a-2)=0$

$a=-3, \quad a=2$

$\boxed{a=2}$

2 The points A, B, C have position vectors

$$4\mathbf{i} - 4\mathbf{j} + \mathbf{k}, \quad -4\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}, \quad 4\mathbf{i} - \mathbf{j} - 2\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax + by + cz = d$. [5]

$\vec{OA} = 4i - 4j + k$ $\qquad \vec{AB} = \vec{OB} - \vec{OA}$

$\vec{OB} = -4i + 3j - 4k$ $\qquad = -4i + 3j - 4k - 4i + 4j - k$

$\vec{OC} = 4i - j - 2k$ $\qquad \vec{AB} = -8i + 7j - 5k$

$\vec{AC} = \vec{OC} - \vec{OA} = \cancel{4i} - j - 2k - \cancel{4i} + 4j - k$

$\vec{AC} = 3j - 3k$

$$\hat{n} = \begin{vmatrix} i & j & k \\ -8 & 7 & -5 \\ 0 & 3 & -3 \end{vmatrix} = i \begin{vmatrix} 7 & -5 \\ 3 & -3 \end{vmatrix} - j \begin{vmatrix} -8 & -5 \\ 0 & -3 \end{vmatrix} + k \begin{vmatrix} -8 & 7 \\ 0 & 3 \end{vmatrix}$$

$= i(-21 + 15) - j(24) + k(-24)$

$= -6i - 24j - 24k \Rightarrow \begin{pmatrix} -6 \\ -24 \\ -24 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}$

$x + 4y + 4z = d$

$d = 4 + 4(-4) + 4(1) = -8$

$\boxed{x + 4y + 4z = -8}$

(b) Find the perpendicular distance from O to the plane ABC. [2]

$\boxed{\frac{8}{\sqrt{1^2+4^2+4^2}} = 1.39}$

(c) The point D has position vector $2\mathbf{i}+3\mathbf{j}-3\mathbf{k}$.

Find the coordinates of the point of intersection of the line OD with the plane ABC. [3]

$r = t\begin{pmatrix}2\\3\\-3\end{pmatrix} \Rightarrow \begin{pmatrix}2t\\3t\\-3t\end{pmatrix}$

$x+4y+4z=-8$

$2t+4(3t)+4(-3t)=-8$

$2t+\cancel{12t}-\cancel{12t}=-8$

$t=-4$

$\Rightarrow \boxed{\begin{pmatrix}-8\\-12\\12\end{pmatrix}}$

3 The sequence of positive numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 > 4$ and, for $n \geqslant 1$,

$$u_{n+1} = \frac{u_n^2 + u_n + 12}{2u_n}.$$

(a) By considering $u_{n+1} - 4$, or otherwise, prove by mathematical induction that $u_n > 4$ for all positive integers n. [5]

for $n=1$

$u_1 > 4$ as given

Suppose true for $n=k$

$$u_{k+1} = \frac{u_k^2 + u_k + 12}{2u_k}$$

Prove true for $n=k+1$

$$u_{k+1} - 4 = \frac{u_k^2 + u_k + 12}{2u_k} - 4$$

$$u_{k+1} - 4 = \frac{u_k^2 + u_k + 12 - 8u_k}{2u_k}$$

$$u_{k+1} - 4 = \frac{u_k^2 - 7u_k + 12}{2u_k}$$

$$u_{k+1} - 4 = \frac{u_k^2 - 4u_k - 3u_k + 12}{2u_k}$$

$$u_{k+1} - 4 = \frac{(u_k - 4)(u_k - 3)}{2u_k}$$

$$\frac{(u_k - 4)(u_k - 3)}{2u_k} > 0$$

$$u_{k+1} - 4 > 0$$

$$\boxed{u_{k+1} > 4}$$

Hence induction is complete.

(b) Show that $u_{n+1} < u_n$ for $n \geqslant 1$. [3]

$$\frac{u_n^2 + u_n + 12}{2u_n} < u_n$$

$$\frac{u_n^2 + u_n + 12}{2u_n} - u_n < 0$$

$$\frac{-u_n^2 + u_n + 12}{2u_n} < 0$$

$$\frac{-(u_n - 4)(u_n + 3)}{2u_n} < 0$$

$u_n > 4$

$$\frac{(u_n - 4)(u_n + 3)}{2u_n} > 0$$

$$\therefore u_{n+1} - u_n < 0$$

$$\boxed{u_{n+1} < u_n}$$

$$-\frac{(u_n - 4)(u_n + 3)}{2u_n} < 0$$

4 The cubic equation $2x^3 + 5x^2 - 6 = 0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\frac{1}{\alpha^3}, \frac{1}{\beta^3}, \frac{1}{\gamma^3}$. [3]

$y = \frac{1}{x^3} \Rightarrow x^{-3} \Rightarrow x = y^{-1/3}$

$2y^{-1} + 5y^{-2/3} - 6 = 0 \Rightarrow 5y^{-2/3} = 6 - 2y^{-1}$

$125y = (6y-2)^3$

$\boxed{216y^3 - 216y^2 - 53y - 8 = 0}$

(b) Find the value of $\frac{1}{\alpha^6} + \frac{1}{\beta^6} + \frac{1}{\gamma^6}$. [3]

$\frac{1}{\alpha^6} + \frac{1}{\beta^6} + \frac{1}{\gamma^6} = \left(\frac{1}{\alpha^3}\right)^2 + \left(\frac{1}{\beta^3}\right)^2 + \left(\frac{1}{\gamma^3}\right)^2$

$= \left(\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3}\right)^2 - 2\left(\frac{1}{\alpha^3\beta^3} + \frac{1}{\alpha^3\gamma^3} + \frac{1}{\beta^3\gamma^3}\right)$

$= (1)^2 - 2\left(\frac{-53}{216}\right)$

$\boxed{\frac{1}{\alpha^6} + \frac{1}{\beta^6} + \frac{1}{\gamma^6} = \frac{161}{108}}$

(c) Find also the value of $\frac{1}{\alpha^9}+\frac{1}{\beta^9}+\frac{1}{\gamma^9}$. [2]

$\frac{1}{\alpha^9}+\frac{1}{\beta^9}+\frac{1}{\gamma^9}=\left(\frac{1}{\alpha^3}\right)^3+\left(\frac{1}{\beta^3}\right)^3+\left(\frac{1}{\gamma^3}\right)^3$

$216y^3-216y^2-53y-8=0$

$216S_3-216S_2-53S_1-24=0$

$S_3=\frac{216\left(\frac{161}{108}\right)+53(1)+24}{216}$

$\boxed{\frac{1}{\alpha^9}+\frac{1}{\beta^9}+\frac{1}{\gamma^9}=\frac{133}{72}}$

5 The curve C has equation $y = \dfrac{2x^2 - x - 1}{x^2 + x + 1}$.

(a) Show that C has no vertical asymptotes and state the equation of the horizontal asymptote of C. [3]

$x^2+x+1=0$ $\boxed{y=2}$ ← HA

$b^2-4ac < 0$

$1^2-4(1)(1) < 0$

$1-4 < 0$

$-3 < 0$

so no real sols

so no VAs.

(b) Find the coordinates of the stationary points on C. [4]

$\dfrac{dy}{dx} = \dfrac{(x^2+x+1)(4x-1) - (2x+1)(2x^2-x-1)}{(x^2+x+1)^2}$

$3x^2+6x=0$

$3x(x+2)=0$

$x=0$ $x+2=0$

$y=-1$ $x=-2$

$y=3$

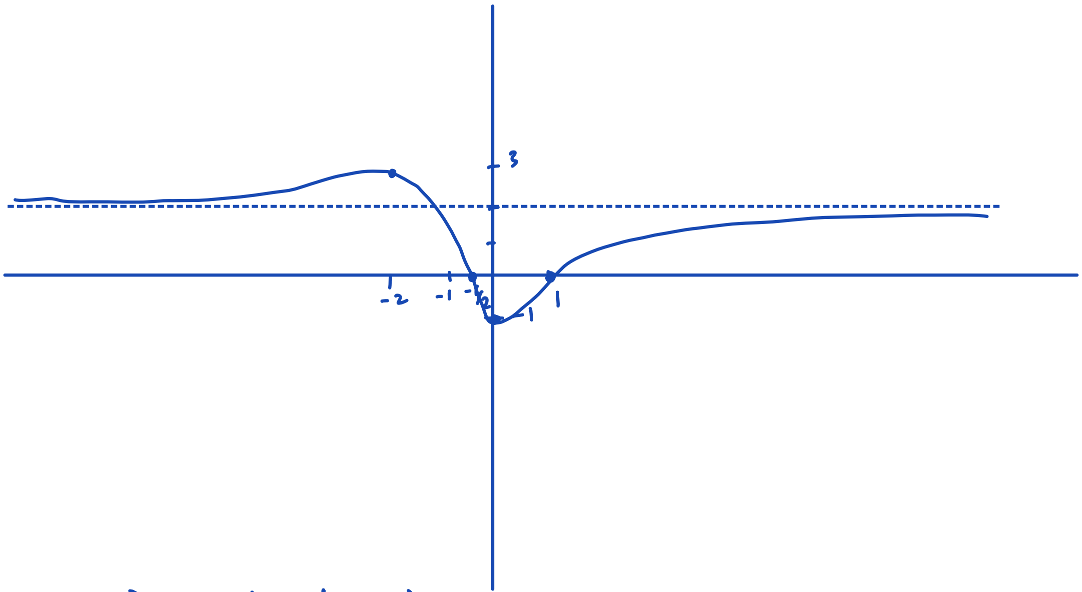
$\boxed{(0,-1) \quad (-2,3)}$

$y = \dfrac{2x^2 - x - 1}{x^2 + x + 1}$.

(0,−1)
(1,0)
$(-\frac{1}{2},0)$

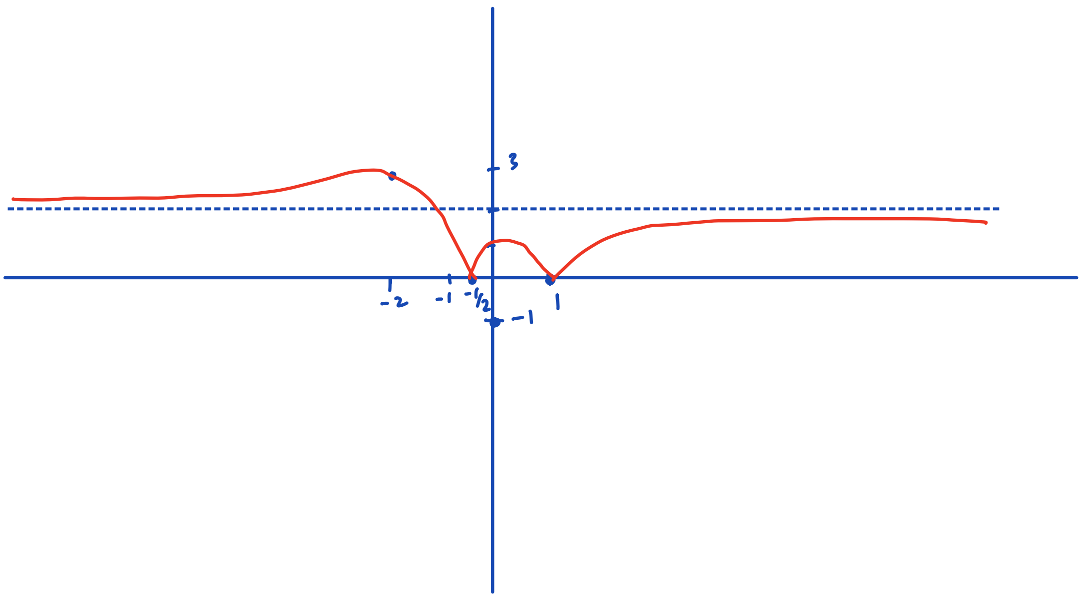
(0,−1) (−2,3)
y=2

(c) Sketch C, stating the coordinates of the intersections with the axes. [3]



(1,0) $(-\frac{1}{2},0)$ (0,−1)

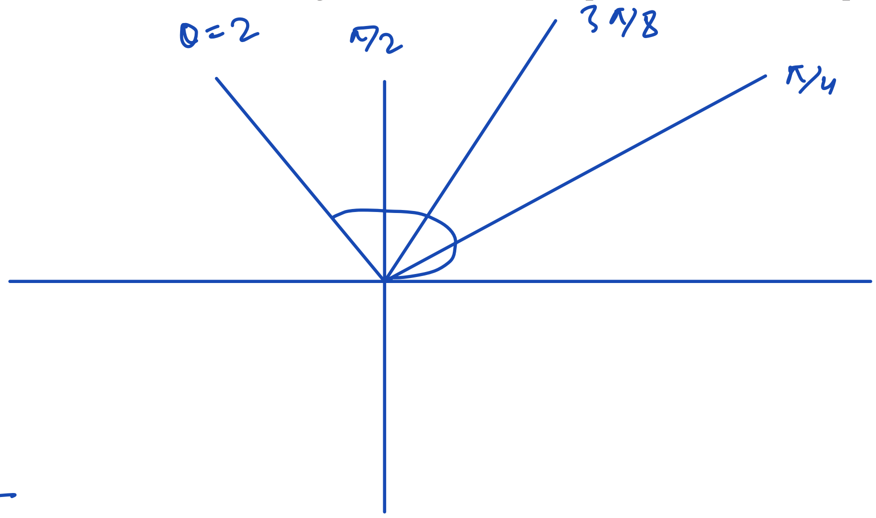
(d) Sketch the curve with equation $y = \left|\dfrac{2x^2 - x - 1}{x^2 + x + 1}\right|$ and state the set of values of k for which $\left|\dfrac{2x^2 - x - 1}{x^2 + x + 1}\right| = k$ has 4 distinct real solutions. [2]



$0 < k < 1$

6 The curve C has polar equation $r^2 = \tan^{-1}\left(\frac{1}{2}\theta\right)$, where $0 \leqslant \theta \leqslant 2$.

(a) Sketch C and state, in exact form, the greatest distance of a point on C from the pole. [3]



$\sqrt{\frac{\pi}{4}}$

(b) Find the exact value of the area of the region bounded by C and the half-line $\theta = 2$. [5]

$\frac{1}{2}\int_0^2 \tan^{-1}\left(\frac{\theta}{2}\right)\, d\theta$

$\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}\, dx$

$u = \tan^{-1}\left(\frac{1}{2}\theta\right)$ $v' = 1$

$u' = \frac{1}{2} \cdot \frac{1}{\frac{\theta^2}{4}+1}$ $v = \theta$

$\frac{1}{2}\left[\theta \tan^{-1}\left(\frac{1}{2}\theta\right) - \int_0^2 \frac{1}{2}\theta \cdot \frac{4}{\theta^2+4}\, d\theta\right]$

$\frac{1}{2}\left[\theta \tan^{-1}\left(\frac{1}{2}\theta\right) - \int_0^2 \frac{2\theta}{\theta^2+4}\, d\theta\right]$

$\frac{1}{2}\left[\theta \tan^{-1}\left(\frac{1}{2}\theta\right) - \ln(\theta^2+4)\right]_0^2$

$\frac{1}{4}\pi - \frac{1}{2}\ln 8 + \frac{1}{2}\ln 4 = \boxed{\frac{1}{4}\pi - \frac{1}{2}\ln 2}$

Now consider the part of C where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(c) Show that, at the point furthest from the half-line $\theta = \frac{1}{2}\pi$,

$$(\theta^2 + 4)\tan^{-1}\left(\tfrac{1}{2}\theta\right)\sin\theta - \cos\theta = 0$$

and verify that this equation has a root between 0.6 and 0.7. [5]

$r^2 = \tan^{-1}\left(\frac{1}{2}\theta\right)$ $\qquad x = r\cos\theta$

$x = \left[\tan^{-1}\left(\frac{1}{2}\theta\right)\right]^{1/2}\cos\theta$ $\qquad r = \frac{x}{\cos\theta}$

$\frac{dx}{d\theta} = -\sin\theta\left(\tan^{-1}\left(\frac{1}{2}\theta\right)\right)^{1/2} + \cos\theta\left(\tan^{-1}\left(\frac{1}{2}\theta\right)\right)^{-1/2} \cdot \left(\theta^2+4\right)^{-1} = 0$

$(\theta^2+4)\tan^{-1}\left(\frac{1}{2}\theta\right)\sin\theta - \cos\theta = 0$

for $0.6 \Rightarrow -0.108$

for $0.7 = 0.209$

sign change so root exists b/w them

7 The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & k & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

(a) Find the set of values of k for which **A** is non-singular. [3]

$\begin{vmatrix} k & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & k \\ 7 & 8 \end{vmatrix} = 0$

$-12k + 60 = 0$

$\boxed{k \neq 5 \quad (k < 5,\ k > 5)}$

(b) Given that **A** is non-singular, find, in terms of k, the entries in the top row of $\mathbf{A}^{-1}$. [4]

$C_{11} = \begin{vmatrix} k & 6 \\ 8 & 9 \end{vmatrix} = 9k - 48$

$C_{12} = -\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = -(18 - 24) = 6$

$C_{13} = \begin{vmatrix} 2 & 3 \\ k & 6 \end{vmatrix} = 12 - 3k$

$A^{-1} = \begin{pmatrix} \frac{9k-48}{60-12k} & \frac{6}{60-12k} & \frac{12-3k}{60-2k} \\ - & - \quad - & - \\ - & - & - \end{pmatrix}$

(c) Given that $\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, give an example of a matrix $\mathbf{C}$ such that $\mathbf{BAC} = \begin{pmatrix} 2 & 1 \\ k & 4 \end{pmatrix}$. [4]

$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & k & 6 \\ 7 & 8 & 9 \end{pmatrix}$

$BA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & k & 6 \\ 7 & 8 & 9 \end{pmatrix}$

$BA = \begin{pmatrix} 1 & 2 & 3 \\ 4 & k & 6 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 & 3 \\ 4 & k & 6 \end{pmatrix} \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix} = \begin{pmatrix} a+2b+3c & d+2e+3f \\ \underline{4a+kb+6c} & 4d+ke+6f \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ k & 4 \end{pmatrix}$

let $b=1$, $a=0, c=0$ $\quad e=0, f=0, d=1$

$\boxed{\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}}$

(d) Find the set of values of k for which the transformation in the x-y plane represented by $\begin{pmatrix} 2 & 1 \\ k & 4 \end{pmatrix}$ has two distinct invariant lines through the origin. [6]

$$\begin{pmatrix} 2 & 1 \\ k & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ kx+4y \end{pmatrix}$$

$$kx + 4mx = m(2x+mx)$$

$$m^2 - 2m - k = 0$$

$$4 + 4k > 0$$

$$\boxed{k > -1}$$

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

BLANK PAGE

BLANK PAGE

BLANK PAGE