1 **(a)** Use standard results from the list of formulae (MF19) to find $\sum_{r=1}^{n}(3r^2+3r+1)$ in terms of n, simplifying your answer. [3]

$3\sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$

$= \frac{3}{6}n(n+1)(2n+1) + \frac{3}{2}n(n+1) + n$

$= \frac{1}{2}n(n+1)(2n+1) + \frac{3}{2}n(n+1) + n$

$= \frac{1}{2}n\left[(n+1)(2n+1) + 3(n+1) + 2\right]$

$= \frac{1}{2}n\left[2n^2+3n+1+3n+3+2\right]$

$= \frac{1}{2}n\left[2n^2+6n+6\right]$

$= n(n^2+3n+3]$

$= n^3+3n^2+3n$

$$\boxed{\sum_{r=1}^{n}(3r^2+3r+1) = n^3+3n^2+3n}$$

(b) Show that

$$\frac{1}{r^3} - \frac{1}{(r+1)^3} = \frac{3r^2+3r+1}{r^3(r+1)^3}$$

and hence use the method of differences to find $\sum_{r=1}^{n} \frac{3r^2+3r+1}{r^3(r+1)^3}$. [5]

$$\frac{(r+1)^3 - r^3}{r^3(r+1)^3} \Rightarrow \frac{\cancel{r^3} + 3r^2 + 3r + 1 - \cancel{r^3}}{r^3(r+1)^3} = \frac{3r^2+3r+1}{r^3(r+1)^3}$$

$$\sum_{r=1}^{n} \frac{1}{r^3} - \frac{1}{(r+1)^3} \Rightarrow 1 - \frac{1}{(n+1)^3}$$

$r=1$: $\frac{1}{1^3} - \cancel{\frac{1}{2^3}}$

$r=2$: $\cancel{\frac{1}{2^3}} - \cancel{\frac{1}{3^3}}$

$r=3$: $\cancel{\frac{1}{3^3}} - \cancel{\frac{1}{4^3}}$

$r=n-1$: $\cancel{\frac{1}{(n-1)^3}} - \cancel{\frac{1}{n^3}}$

$r=n$: $\cancel{\frac{1}{n^3}} - \frac{1}{(n+1)^3}$

(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{3r^2+3r+1}{r^3(r+1)^3}$. [1]

$\frac{1}{(n+1)^3} \to 0$ as $n \to \infty$

$$\lim_{n\to\infty} \left(1 - \frac{1}{(n+1)^3}\right)$$

$\therefore$ $\boxed{\lim_{n\to\infty} \Rightarrow 1}$

2 Prove by mathematical induction that, for all positive integers n,

$$\frac{d^n}{dx^n}(x^2e^x) = (x^2 + 2nx + n(n-1))e^x.$$ [6]

for $n=1$

$\frac{d}{dx}(x^2e^x) = 2xe^x + x^2e^x$ (LHS)

$(x^2 + 2x + 1(\cancel{1}-1))e^x = (x^2+2x)e^x = 2xe^x + x^2e^x$ (RHS)

LHS = RHS

Suppose true for $n=k$

$\frac{d^k}{dx^k}(x^2e^x) = (x^2 + 2kx + k(k-1))e^x$

Prove true for $n=k+1$

$\frac{d^{k+1}}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^k}{dx^k}(x^2e^x)\right)$

$= \frac{d}{dx}\left[(x^2+2kx+k^2-k)e^x\right]$

$= (2x+2k)e^x + e^x(x^2+2kx+k^2-k)$

$= e^x(2x+2k+x^2+2kx+k^2-k)$

$= (x^2+(2k+2)x+k^2+k)e^x$

$= x^2 + 2(k+1)x + (k+1)k\ )e^x$

Hence induction is complete.

3 The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, where k is a constant and $k \neq 0$ and $k \neq 1$.

(a) The matrix $\mathbf{M}$ represents a sequence of two geometrical transformations. State the type of each transformation, and make clear the order in which they are applied. [2]

Shear followed by a stretch

The unit square in the $x-y$ plane is transformed by $\mathbf{M}$ onto parallelogram $OPQR$.

(b) Find, in terms of k, the area of parallelogram $OPQR$ and the matrix which transforms $OPQR$ onto the unit square. [3]

$M = \begin{pmatrix} k & 0 \\ 1 & 1 \end{pmatrix}$ $\det(M) = k$ $\boxed{|k| = |OPQR|}$

$M^{-1} = \frac{1}{k}\begin{pmatrix} 1 & 0 \\ -1 & k \end{pmatrix}$

(c) Show that the line through the origin with gradient $\dfrac{1}{k-1}$ is invariant under the transformation in the $x-y$ plane represented by $\mathbf{M}$. [3]

$\begin{pmatrix} k & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{pmatrix} k & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ \frac{1}{k-1}x \end{pmatrix} = \begin{pmatrix} kx \\ x + \frac{1}{k-1}x \end{pmatrix} = \begin{pmatrix} kx \\ \frac{k}{k-1}x \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ \frac{1}{k-1}x \end{pmatrix}$

$\begin{pmatrix} kx \\ \frac{k}{k-1}x \end{pmatrix} = k\begin{pmatrix} x \\ \frac{1}{k-1}x \end{pmatrix}$

4 The cubic equation $27x^3 + 18x^2 + 6x - 1 = 0$ has roots α, β, γ.

(a) Show that a cubic equation with roots $3\alpha + 1$, $3\beta + 1$, $3\gamma + 1$ is

$$y^3 - y^2 + y - 2 = 0.$$ [3]

$y = 3x + 1$

$\frac{y-1}{3} = x$

$27\left(\frac{y-1}{3}\right)^3 + 18\left(\frac{y-1}{3}\right)^2 + 6\left(\frac{y-1}{3}\right) - 1 = 0$

$\frac{27}{27}(y-1)^3 + \frac{18}{9}(y-1)^2 + 2(y-1) - 1 = 0$

$(y-1)^3 + 2(y-1)^2 + 2(y-1) - 1 = 0$

$y^3 - 3y^2 + 3y - 1 + 2y^2 - 4y + 2 + 2y - 3 = 0$

$\boxed{y^3 - y^2 + y - 2 = 0}$

The sum $(3\alpha+1)^n+(3\beta+1)^n+(3\gamma+1)^n$ is denoted by S_n.

(b) Find the values of S_2 and S_3. [4]

$S_2 = \sum \alpha - 2\sum\alpha\beta$

$= 1 - 2(1)$

$= -1$

$\boxed{S_2 = -1}$

$S_3 - S_2 + S_1 - 6 = 0$

$S_3 = S_2 + 6 - S_1$

$= -1 + 6 - 1$

$= 4$

$\boxed{S_3 = 4}$

$y^3 - y^2 + y - 2 = 0$

(c) Find the values of S_{-1} and S_{-2}. [3]

$S_{-1} = \dfrac{\sum\alpha\beta}{\sum\alpha\beta\gamma} = \dfrac{1}{-2}$

$\boxed{S_{-1} = \dfrac{(3\alpha+1)(3\beta+1)+(3\beta+1)(3\gamma+1)+(3\alpha+1)(3\gamma+1)}{(3\alpha+1)(3\beta+1)(3\gamma+1)} = -\dfrac{1}{2}}$

divide by y^2

$y - 1 + y^{-1} - 2y^{-2} = 0$

$S_1 - 3 + S_{-1} - 2S_{-2} = 0$

$S_{-2} = \dfrac{S_1 + S_{-1} - 3}{2}$

$\boxed{S_{-2} = -\dfrac{3}{4}}$

5 The plane Π_1 has equation $\mathbf{r} = \mathbf{i} - \mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) + \mu(3\mathbf{i} - \mathbf{k})$.

(a) Find an equation for Π_1 in the form $ax + by + cz = d$. [4]

$\hat{n} = \begin{vmatrix} i & j & k \\ 1 & -2 & -3 \\ 3 & 0 & -1 \end{vmatrix} = i\begin{vmatrix} -2 & -3 \\ 0 & -1 \end{vmatrix} - j\begin{vmatrix} 1 & -3 \\ 3 & -1 \end{vmatrix} + k\begin{vmatrix} 1 & -2 \\ 3 & 0 \end{vmatrix}$

$= i - 4j + 3k = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}$

$x - 4y + 3z = d$ $\quad (1, -1, -2)$

$1 - 4(-1) + 3(-2) = d$

$d = -1$

$\boxed{x - 4y + 3z = -1}$

The line l, which does not lie in Π_1, has equation $\mathbf{r} = -3\mathbf{i} + \mathbf{k} + t(\mathbf{i} + \mathbf{j} + \mathbf{k})$.

(b) Show that l is parallel to Π_1. [2]

$\begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 - 4 + 3 = 0$

So parallel.

(c) Find the distance between l and Π_1. [3]

$|b_1 \times b_2| = \sqrt{1^2 + (-4)^2 + 3^2} = \sqrt{26}$

$\overrightarrow{PQ} = -3i + k - i + j + 2k \Rightarrow -4i + j + 3k = \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}$

$\frac{1}{\sqrt{26}}\left[\begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}\right] = \boxed{\frac{1}{\sqrt{26}} = 0.196}$

(d) The plane Π_2 has equation $3x + 3y + 2z = 1$. $x - 4y + 3z = -1$

Find a vector equation of the line of intersection of Π_1 and Π_2. [4]

$\hat{n} = \begin{vmatrix} i & j & k \\ 1 & -4 & 3 \\ 3 & 3 & 2 \end{vmatrix} = i\begin{vmatrix} -4 & 3 \\ 3 & 2 \end{vmatrix} - j\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} + k\begin{vmatrix} 1 & -4 \\ 3 & 3 \end{vmatrix}$

$= \begin{pmatrix} -17 \\ 7 \\ 15 \end{pmatrix}$

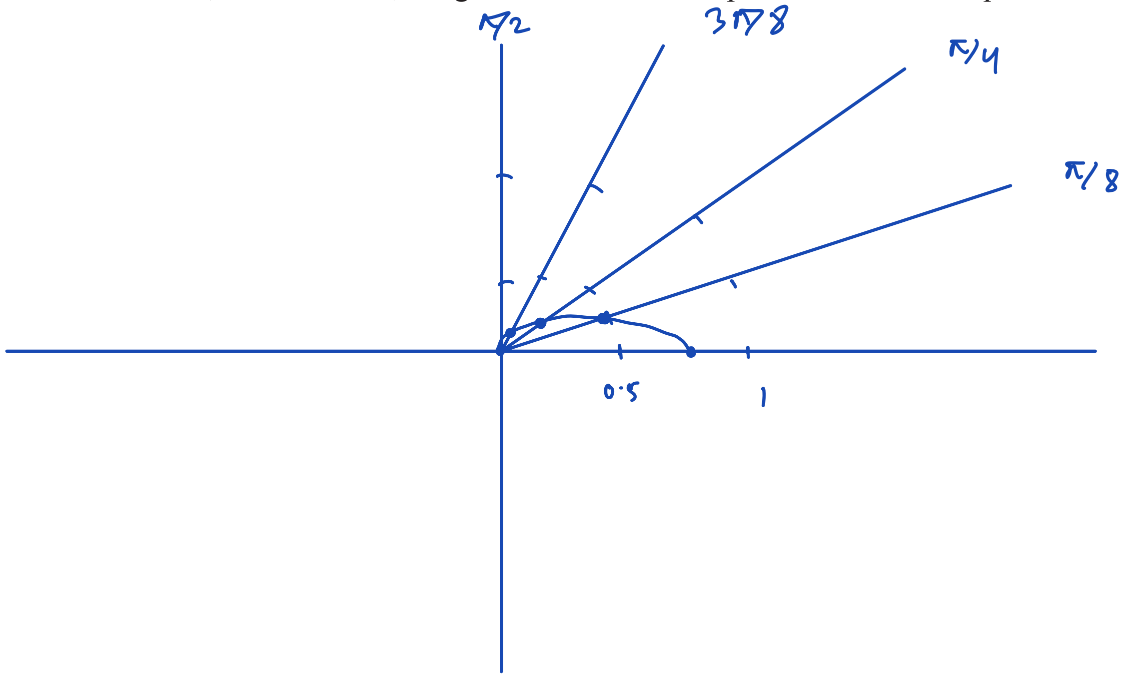
let $x = 0$ $\quad 3y + 2z = 1 \qquad y = 5/17$

$\quad -4y + 3z = -1 \qquad z = 1/17 \Rightarrow \begin{pmatrix} 0 \\ 5/17 \\ 1/17 \end{pmatrix}$

$r = \begin{pmatrix} 0 \\ 5/17 \\ 1/17 \end{pmatrix} + \lambda\begin{pmatrix} -17 \\ 7 \\ 15 \end{pmatrix}$

6 The curve C has polar equation $r = e^{-\theta} - e^{-\frac{1}{2}\pi}$, where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) Sketch C and state, in exact form, the greatest distance of a point on C from the pole. [3]



$1 - e^{-\frac{1}{2}\pi}$

(b) Find the exact value of the area of the region bounded by C and the initial line. [5]

$\frac{1}{2}\int_0^{\pi/2} (e^{-\theta} - e^{-\frac{1}{2}\pi})^2 \, d\theta$

$\frac{1}{2}\int_0^{\pi/2} e^{-2\theta} - 2e^{-\theta-\frac{1}{2}\pi} + e^{-\pi} \, d\theta$

$\frac{1}{2}\left[-\frac{1}{2}e^{-2\theta} + 2e^{-\theta-\frac{1}{2}\pi} + e^{-\pi}\theta \right]_0^{\pi/2}$

$= \frac{1}{2}\left(-\frac{1}{2}e^{-\pi} + 2e^{-\pi} + \frac{1}{2}\pi e^{-\pi} + \frac{1}{2} - 2e^{-\pi/2} \right) =$

$\Rightarrow \boxed{\frac{3}{4}e^{-\pi} + \frac{1}{4}\pi e^{-\pi} - e^{-\pi/2} + \frac{1}{4}}$

(c) Show that, at the point on C furthest from the initial line,

$$1 - e^{\theta - \frac{1}{2}\pi} - \tan\theta = 0$$

and verify that this equation has a root between 0.56 and 0.57. [5]

$r = e^{-\theta} - e^{-\pi/2}$

$y = r\sin\theta$

$r = \frac{y}{\sin\theta}$

$y = (e^{-\theta} - e^{-\pi/2})\sin\theta$

$\frac{dy}{d\theta} = \cos\theta(e^{-\theta} - e^{-\pi/2}) + \sin\theta(-e^{-\theta})$

$= \frac{\cancel{\cos\theta} e^{-\theta}}{\cancel{\cos\theta} e^{-\theta}} - \frac{\cancel{\cos\theta} e^{-\pi/2}}{\cancel{\cos\theta} e^{-\theta}} - \frac{\sin\theta e^{-\theta}}{\cos\theta e^{-\theta}}$

$= 1 - e^{-\pi/2 + \theta} - \tan\theta = 0$

$1 - e^{\theta - \pi/2} - \tan\theta = 0$

$1 - e^{0.50 - \pi/2} - \tan 0.56 = 0.00912$

$1 - e^{0.57 - \pi/2} - \tan 0.57 = -0.00856$

sign change

7 The curve C has equation $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \dfrac{x^2}{x+1}$.

(a) Find the equations of the asymptotes of C. [3]

$x+1=0$

$\boxed{x=-1} \leftarrow$ VA

$$\begin{array}{r} x-1 \\ x+1 \overline{)\, x^2 \qquad} \\ -\underline{(x^2+x)} \\ -x \\ -\underline{(-x-1)} \\ 1 \end{array}$$

$\boxed{y = x-1} \leftarrow$ OA

(b) Find the coordinates of any stationary points on C. [2]

$u = x^2 \quad v = x+1$

$u' = 2x \quad v' = 1$

$\dfrac{dy}{dx} = \dfrac{2x(x+1) - x^2}{(x+1)^2}$

$x=0$	$x=-2$
$y=0$	$y=-4$

$2x^2 + 2x - x^2 = 0$

$x^2 + 2x = 0$

$x(x+2) = 0$

$x = 0 \quad x+2=0$

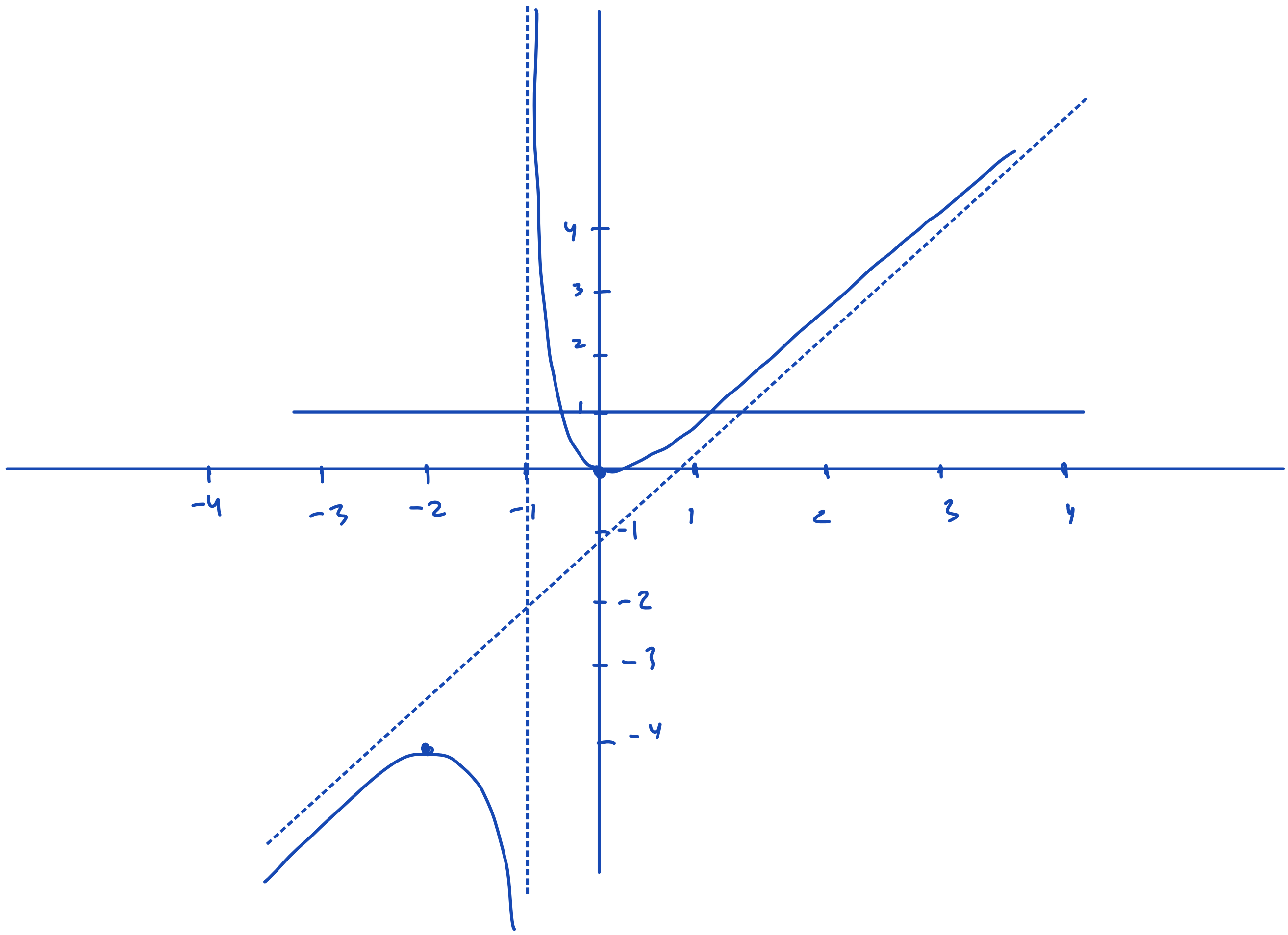
$x = -2$

(0,0) (-2,-4)

y=x-1 x=-1

y=2

(c) Sketch C. [3]



(d) Find the coordinates of any stationary points on the curve with equation $y = \frac{1}{f(x)}$. [2]

$y = \frac{x+1}{x^2}$

$yx^2 = x+1$

$yx^2 - x - 1 = 0$

$b^2 - 4ac$

$(-1)^2 - 4(y)(-1) = 0$

$1 + 4y = 0$

$y = -\frac{1}{4}$

$\underline{x = -2}$

$\boxed{\left(-2, -\frac{1}{4}\right)}$

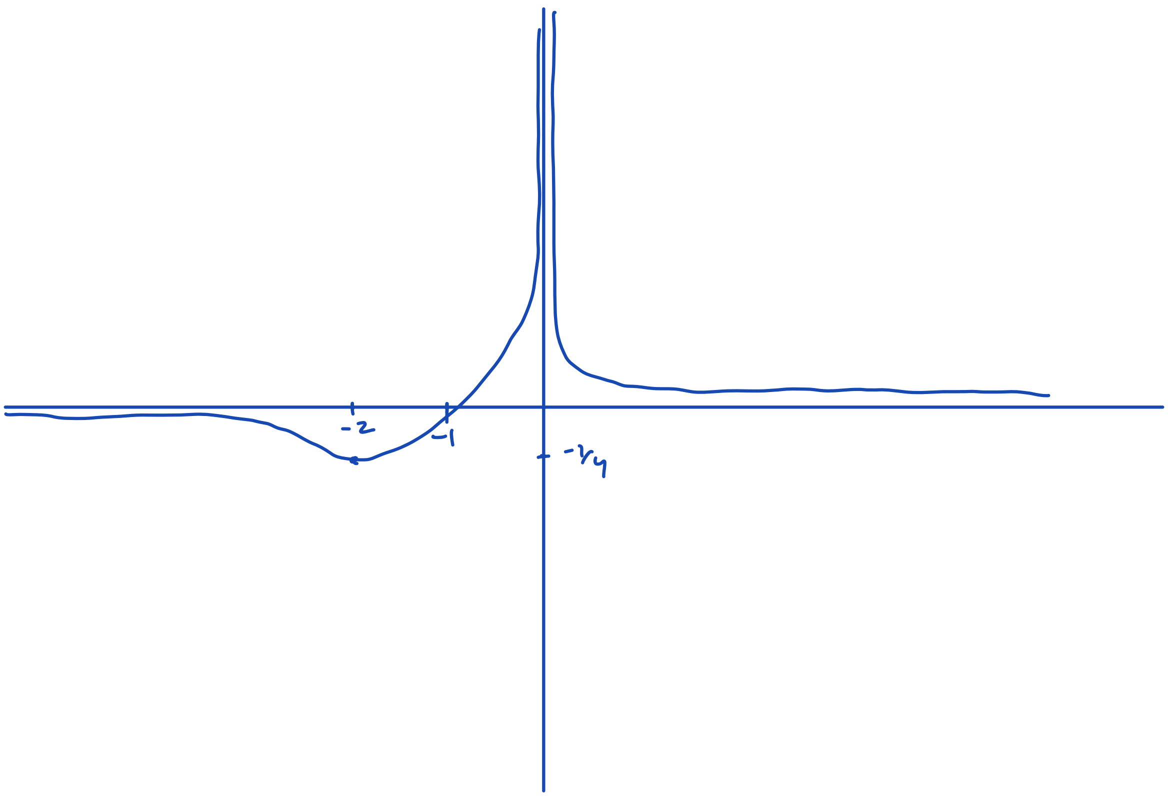
$y = \frac{x+1}{x^2}$

$y = 0$
$x = 0$
A_1

(e) Sketch the curve with equation $y = \frac{1}{\mathrm{f}(x)}$ and find, in exact form, the set of values for which $\frac{1}{\mathrm{f}(x)} > \mathrm{f}(x)$. [6]

$(-1, 0)$

$(-2, -\frac{1}{4})$



$f^2(x) < 1$

$\left(\frac{x^2}{x+1}\right)^2 = 1$

$\frac{x^2}{x+1} = \pm\sqrt{1}$

$\frac{x^2}{x+1} = \pm 1$

ADDITIONAL SHEET

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

$\frac{x^2}{x+1} = 1$ $\frac{x^2}{x+1} = -1$

$x^2 + x + 1 = 0$

$x^2 - x - 1 = 0$ NO SOL

$x = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$

$\boxed{\begin{array}{c} \frac{1}{2} - \frac{1}{2}\sqrt{5} < x < \frac{1}{2} + \frac{1}{2}\sqrt{5} \\ x < -1 \end{array}}$