

- 1 (a) Use standard results from the list of formulae (MF19) to find $\sum_{r=1}^n (3r^2 + 3r + 1)$ in terms of n , simplifying your answer. [3]

$$3 \sum_{r=1}^n r^2 + 3 \sum_{r=1}^n r + \sum_{r=1}^n 1$$

$$= \frac{3}{6} n(n+1)(2n+1) + \frac{3}{2} n(n+1) + n$$

$$= \frac{1}{2} n(n+1)(2n+1) + \frac{3}{2} n(n+1) + n$$

$$= \frac{1}{2} n \left[(n+1)(2n+1) + 3(n+1) + 2 \right]$$

$$= \frac{1}{2} n \left[2n^2 + 3n + 1 + 3n + 3 + 2 \right]$$

$$= \frac{1}{2} n \left[2n^2 + 6n + 6 \right]$$

$$= n(n^2 + 3n + 3)$$

$$= n^3 + 3n^2 + 3n$$

$$\sum_{r=1}^n (3r^2 + 3r + 1) = n^3 + 3n^2 + 3n$$

(b) Show that

$$\frac{1}{r^3} - \frac{1}{(r+1)^3} = \frac{3r^2 + 3r + 1}{r^3(r+1)^3}$$

and hence use the method of differences to find $\sum_{r=1}^n \frac{3r^2 + 3r + 1}{r^3(r+1)^3}$. [5]

$$\frac{(r+1)^3 - r^3}{r^3(r+1)^3} \Rightarrow \frac{\cancel{r^3} + 3r^2 + 3r + 1 - \cancel{r^3}}{r^3(r+1)^3} = \frac{3r^2 + 3r + 1}{r^3(r+1)^3}$$

$$\sum_{r=1}^n \left(\frac{1}{r^3} - \frac{1}{(r+1)^3} \right) \Rightarrow 1 - \frac{1}{(n+1)^3}$$

$$\begin{array}{l} r=1: \quad \frac{1}{1^3} - \frac{1}{2^3} \\ r=2: \quad \frac{1}{2^3} - \frac{1}{3^3} \\ r=3: \quad \frac{1}{3^3} - \frac{1}{4^3} \\ r=n-1: \quad \frac{1}{(n-1)^3} - \frac{1}{n^3} \\ r=n: \quad \frac{1}{n^3} - \frac{1}{(n+1)^3} \end{array}$$

(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{3r^2 + 3r + 1}{r^3(r+1)^3}$. [1]

$$\lim_{n \rightarrow \infty} \left(1 - \frac{1}{(n+1)^3} \right) \quad \frac{1}{(n+1)^3} \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\therefore \lim_{n \rightarrow \infty} \Rightarrow 1$$

2 Prove by mathematical induction that, for all positive integers n ,

$$\frac{d^n}{dx^n}(x^2 e^x) = (x^2 + 2nx + n(n-1))e^x. \quad [6]$$

for $n=1$

$$\frac{d}{dx}(x^2 e^x) = 2xe^x + x^2 e^x \quad (\text{LHS})$$

$$(x^2 + 2x + 1(\cancel{x-1}))e^x = (x^2 + 2x)e^x = 2xe^x + x^2 e^x \quad (\text{RHS})$$

$$\text{LHS} = \text{RHS}$$

Suppose true for $n=k$

$$\frac{d^k}{dx^k}(x^2 e^x) = (x^2 + 2kx + k(k-1))e^x$$

Prove true for $n=k+1$

$$\frac{d^{k+1}}{dx^{k+1}} = \frac{d}{dx} \left(\frac{d^k}{dx^k}(x^2 e^x) \right)$$

$$= \frac{d}{dx} \left[(x^2 + 2kx + k^2 - k)e^x \right]$$

$$= (2x + 2k)e^x + e^x(x^2 + 2kx + k^2 - k)$$

$$= e^x(2x + 2k + x^2 + 2kx + k^2 - k)$$

$$= (x^2 + (2k+2)x + k^2 + k)e^x$$

$$= (x^2 + 2(k+1)x + (k+1)k)e^x$$

Hence induction is complete.

3 The matrix \mathbf{M} is given by $\mathbf{M} = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, where k is a constant and $k \neq 0$ and $k \neq 1$.

- (a) The matrix \mathbf{M} represents a sequence of two geometrical transformations. State the type of each transformation, and make clear the order in which they are applied. [2]

Shear followed by a stretch

The unit square in the x - y plane is transformed by \mathbf{M} onto parallelogram $OPQR$.

- (b) Find, in terms of k , the area of parallelogram $OPQR$ and the matrix which transforms $OPQR$ onto the unit square. [3]

$$\mathbf{M} = \begin{pmatrix} k & 0 \\ 1 & 1 \end{pmatrix} \quad \det(\mathbf{M}) = k \quad \boxed{|k| = |OPQR|}$$

$$\mathbf{M}^{-1} = \frac{1}{k} \begin{pmatrix} 1 & 0 \\ -1 & k \end{pmatrix}$$

- (c) Show that the line through the origin with gradient $\frac{1}{k-1}$ is invariant under the transformation in the x - y plane represented by \mathbf{M} . [3]

$$\begin{pmatrix} k & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} k & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ \frac{1}{k-1}x \end{pmatrix} = \begin{pmatrix} kx \\ x + \frac{1}{k-1}x \end{pmatrix} = \begin{pmatrix} kx \\ \frac{k}{k-1}x \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ \frac{1}{k-1}x \end{pmatrix}$$

$$\begin{pmatrix} kx \\ \frac{k}{k-1}x \end{pmatrix} = k \begin{pmatrix} x \\ \frac{1}{k-1}x \end{pmatrix}$$

4 The cubic equation $27x^3 + 18x^2 + 6x - 1 = 0$ has roots α, β, γ .

(a) Show that a cubic equation with roots $3\alpha + 1, 3\beta + 1, 3\gamma + 1$ is

$$y^3 - y^2 + y - 2 = 0. \quad [3]$$

$$y = 3x + 1$$

$$\frac{y-1}{3} = x$$

$$27\left(\frac{y-1}{3}\right)^3 + 18\left(\frac{y-1}{3}\right)^2 + 6\left(\frac{y-1}{3}\right) - 1 = 0$$

$$\frac{27}{27}(y-1)^3 + \frac{18}{9}(y-1)^2 + 2(y-1) - 1 = 0$$

$$(y-1)^3 + 2(y-1)^2 + 2(y-1) - 1 = 0$$

$$y^3 - 3y^2 + 3y - 1 + 2y^2 - 4y + 2 + 2y - 3 = 0$$

$$y^3 - y^2 + y - 2 = 0$$

The sum $(3\alpha+1)^n + (3\beta+1)^n + (3\gamma+1)^n$ is denoted by S_n .

- (b) Find the values of S_2 and S_3 . [4]

$$S_2 = \sum \alpha^2 - 2 \sum \alpha\beta$$

$$= 1 - 2(1)$$

$$= -1$$

$$\boxed{S_2 = -1}$$

$$S_3 - S_2 + S_1 - 6 = 0$$

$$S_3 = S_2 + 6 - S_1$$

$$= -1 + 6 - 1$$

$$= 4$$

$$\boxed{S_3 = 4}$$

$$y^3 - y^2 + y - 2 = 0$$

- (c) Find the values of S_{-1} and S_{-2} . [3]

$$S_{-1} = \frac{\sum \alpha\beta}{\sum \alpha\beta\gamma} = \frac{1}{-2}$$

$$\boxed{S_{-1} = \frac{(3\alpha+1)(3\beta+1) + (3\beta+1)(3\gamma+1) + (3\gamma+1)(3\alpha+1)}{(3\alpha+1)(3\beta+1)(3\gamma+1)} = -\frac{1}{2}}$$

divide by y^2

$$y - 1 + y^{-1} - 2y^{-2} = 0$$

$$S_1 - 3 + S_{-1} - 2S_{-2} = 0$$

$$S_{-2} = \frac{S_1 + S_{-1} - 3}{2}$$

$$\boxed{S_{-2} = -\frac{3}{4}}$$

5 The plane Π_1 has equation $\mathbf{r} = \mathbf{i} - \mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) + \mu(3\mathbf{i} - \mathbf{k})$.

(a) Find an equation for Π_1 in the form $ax + by + cz = d$.

[4]

$$\begin{aligned} \vec{n} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -3 \\ 3 & 0 & -1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -2 & -3 \\ 0 & -1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & -3 \\ 3 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & -2 \\ 3 & 0 \end{vmatrix} \\ &= \mathbf{i} - 4\mathbf{j} + 3\mathbf{k} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} x - 4y + 3z &= d & (1, -1, -2) \\ 1 - 4(-1) + 3(-2) &= d \\ d &= -1 \end{aligned}$$

$$\boxed{x - 4y + 3z = -1}$$

The line l , which does not lie in Π_1 , has equation $\mathbf{r} = -3\mathbf{i} + \mathbf{k} + t(\mathbf{i} + \mathbf{j} + \mathbf{k})$.

(b) Show that l is parallel to Π_1 .

[2]

$$\begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 - 4 + 3 = 0$$

so parallel.

- (c) Find the distance between
- l
- and
- Π_1
- .

[3]

$$|b_1 \times b_2| = \sqrt{1^2 + (-4)^2 + 3^2} = \sqrt{26}$$

$$\vec{PQ} = -3i + k - i + j + 2k \Rightarrow -4i + j + 3k = \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}$$

$$\frac{1}{\sqrt{26}} \left[\begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix} \right] = \boxed{\frac{1}{\sqrt{26}} = 0.196}$$

- (d) The plane
- Π_2
- has equation
- $3x + 3y + 2z = 1$
- .

$$x - 4y + 3z = -1$$

Find a vector equation of the line of intersection of Π_1 and Π_2 .

[4]

$$\vec{n} = \begin{vmatrix} i & j & k \\ 1 & -4 & 3 \\ 3 & 3 & 2 \end{vmatrix} = i \begin{vmatrix} -4 & 3 \\ 3 & 2 \end{vmatrix} - j \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} + k \begin{vmatrix} 1 & -4 \\ 3 & 3 \end{vmatrix}$$

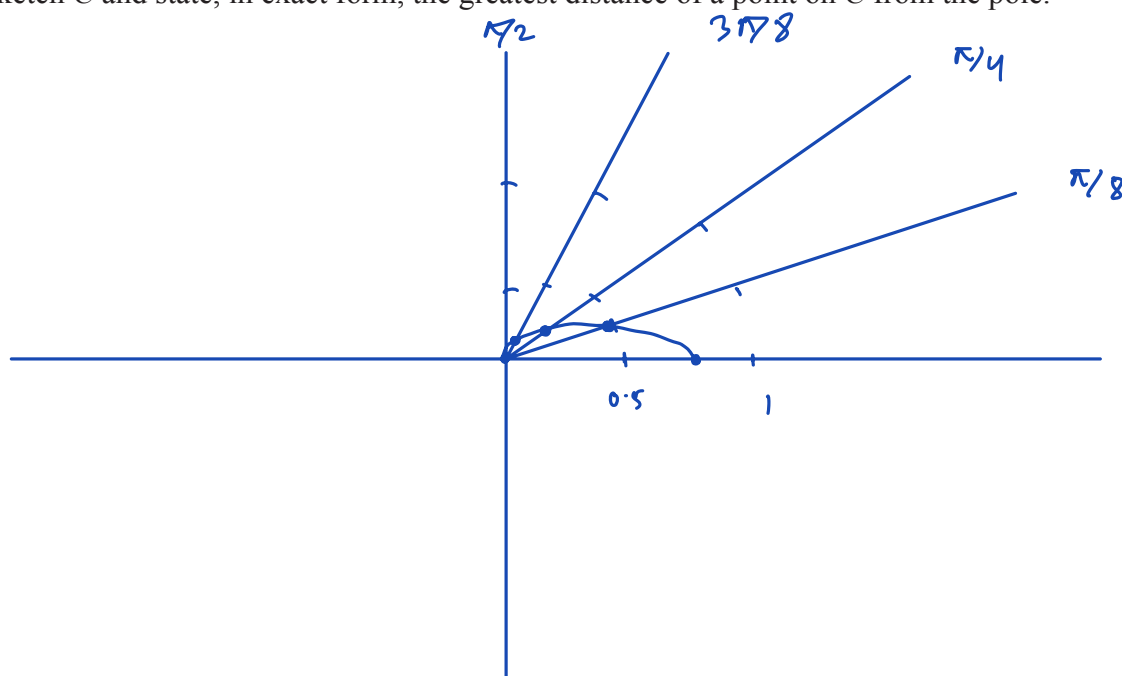
$$= \begin{pmatrix} -17 \\ 7 \\ 15 \end{pmatrix}$$

$$\text{let } x=0 \quad \begin{aligned} 3y + 2z &= 1 \\ -4y + 3z &= -1 \end{aligned} \quad \begin{aligned} y &= 5/17 \\ z &= 1/17 \end{aligned} \Rightarrow \begin{pmatrix} 0 \\ 5/17 \\ 1/17 \end{pmatrix}$$

$$r = \begin{pmatrix} 0 \\ 5/17 \\ 1/17 \end{pmatrix} + \lambda \begin{pmatrix} -17 \\ 7 \\ 15 \end{pmatrix}$$

- 6 The curve C has polar equation $r = e^{-\theta} - e^{-\frac{1}{2}\pi}$, where $0 \leq \theta \leq \frac{1}{2}\pi$.

(a) Sketch C and state, in exact form, the greatest distance of a point on C from the pole. [3]



$$1 - e^{-\frac{1}{2}\pi}$$

(b) Find the exact value of the area of the region bounded by C and the initial line. [5]

$$\frac{1}{2} \int_0^{\pi/2} (e^{-\theta} - e^{-\frac{1}{2}\pi})^2 d\theta$$

$$\frac{1}{2} \int_0^{\pi/2} e^{-2\theta} - 2e^{-\theta - \frac{1}{2}\pi} + e^{-\pi} d\theta$$

$$\frac{1}{2} \left[-\frac{1}{2}e^{-2\theta} + 2e^{-\theta - \frac{1}{2}\pi} + e^{-\pi}\theta \right]_0^{\pi/2}$$

$$= \frac{1}{2} \left(-\frac{1}{2}e^{-\pi} + 2e^{-\pi} + \frac{1}{2}\pi e^{-\pi} + \frac{1}{2} - 2e^{-\pi/2} \right) =$$

$$\Rightarrow \boxed{\frac{3}{4}e^{-\pi} + \frac{1}{4}\pi e^{-\pi} - e^{-\pi/2} + \frac{1}{4}}$$

- (c) Show that, at the point on C furthest from the initial line,

$$r = e^{-\theta} - e^{-\pi/2}$$

$$1 - e^{\theta - \frac{1}{2}\pi} - \tan \theta = 0$$

and verify that this equation has a root between 0.56 and 0.57.

[5]

$$y = (e^{-\theta} - e^{-\pi/2}) \sin \theta$$

$$y = r \sin \theta$$

$$r = \frac{y}{\sin \theta}$$

$$\begin{aligned} \frac{dy}{d\theta} &= \cos \theta (e^{-\theta} - e^{-\pi/2}) + \sin \theta (-e^{-\theta}) \\ &= \frac{\cancel{\cos \theta} e^{-\theta}}{\cancel{\cos \theta} e^{-\theta}} - \frac{\cancel{\cos \theta} e^{-\pi/2}}{\cancel{\cos \theta} e^{-\theta}} - \frac{\sin \theta e^{-\theta}}{\cos \theta e^{-\theta}} \end{aligned}$$

$$= 1 - e^{-\pi/2 + \theta} - \tan \theta = 0$$

$$1 - e^{\theta - \pi/2} - \tan \theta = 0$$

$$1 - e^{0.56 - \pi/2} - \tan 0.56 = 0.00912$$

$$1 - e^{0.57 - \pi/2} - \tan 0.57 = -0.00856$$

↑
sign change

- 7 The curve C has equation $y = f(x)$, where $f(x) = \frac{x^2}{x+1}$.

(a) Find the equations of the asymptotes of C .

[3]

$$x+1=0$$

$$\boxed{x = -1} \leftarrow \text{VA}$$

$$\begin{array}{r} x-1 \\ x+1 \overline{) x^2} \\ \underline{-x^2+x} \\ -x \\ \underline{-x-1} \\ 1 \end{array}$$

$$\boxed{y = x-1} \leftarrow \text{OA}$$

(b) Find the coordinates of any stationary points on C .

[2]

$$u = x^2 \quad v = x+1$$

$$u' = 2x \quad v' = 1$$

$$\frac{dy}{dx} = \frac{2x(x+1) - x^2}{(x+1)^2}$$

$$\boxed{x=0 \quad x=-2}$$

$$\boxed{y=0 \quad y=-4}$$

$$2x^2 + 2x - x^2 = 0$$

$$x^2 + 2x = 0$$

$$x(x+2) = 0$$

$$x=0 \quad x+2=0$$

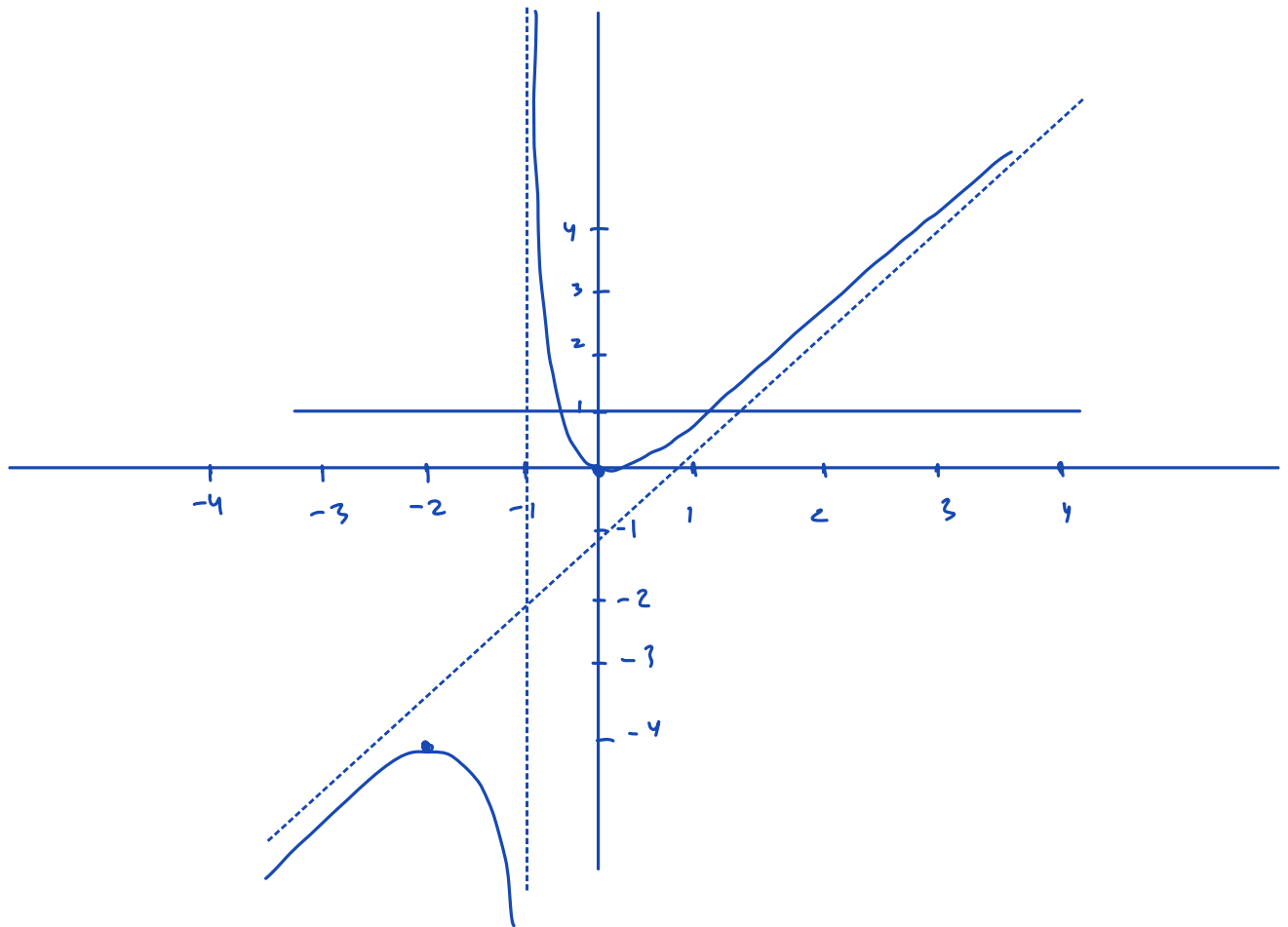
$$x = -2$$

$$(0,0) \quad (-2,-4) \quad y=x-1 \quad x=-1$$

(c) Sketch C.

[3]

$$y=2$$

(d) Find the coordinates of any stationary points on the curve with equation $y = \frac{1}{f(x)}$.

[2]

$$y = \frac{x+1}{x^2}$$

$$x = -2$$

$$yx^2 = x+1$$

$$yx^2 - x - 1 = 0$$

$$b^2 - 4ac$$

$$(-1)^2 - 4(y)(-1) = 0$$

$$1 + 4y = 0$$

$$y = -\frac{1}{4}$$

$$(-2, -\frac{1}{4})$$

$$y = \frac{x+1}{x^2}$$

$$y = 0$$

$$x = 0$$

$$A_1$$

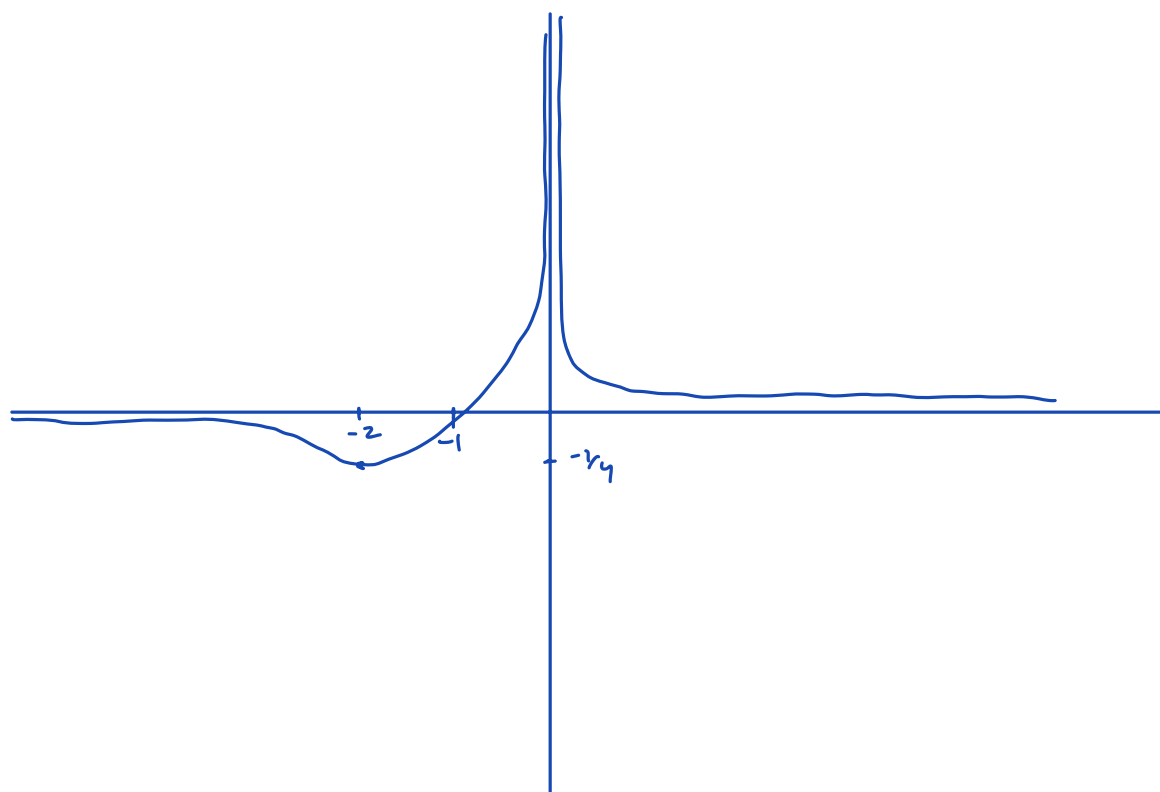
- (e) Sketch the curve with equation $y = \frac{1}{f(x)}$ and find, in exact form, the set of values for which

$$\frac{1}{f(x)} > f(x).$$

[6]

$$(-1, 0)$$

$$(-2, -\frac{1}{2})$$



$$f^2(x) < 1$$

$$\left(\frac{x^2}{x+1}\right)^2 < 1$$

$$\frac{x^2}{x+1} = \pm 1$$

$$\frac{x^2}{x+1} = \pm 1$$

ADDITIONAL SHEET

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

$$\frac{x^2}{x+1} = 1$$

$$\frac{x^2}{x+1} = -1$$

$$x^2 + x + 1 = 0$$

$$x^2 - x - 1 = 0$$

NO SOL

$$x = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$$

$$\frac{1}{2} - \frac{1}{2}\sqrt{5} < x < \frac{1}{2} + \frac{1}{2}\sqrt{5}$$

$$x < -1$$

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