1 **(a)** By considering $(r+1)^2 - r^2$, use the method of differences to prove that

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1).$$ [4]

$\sum_{r=1}^{n} (r+1)^2 - r^2$

$(r+1)^2 - r^2 = \cancel{r^2} + 2r + 1 - \cancel{r^2}$

$\Rightarrow 2r$

$r=1: \quad 2^2 - 1^2$

$r=2: \quad 3^2 - 2^2$

$r=3: \quad 4^2 - 3^2$

$r=n-1 \quad n^2 - (n-1)^2$

$r=n \quad (n+1)^2 - n^2$

$\Rightarrow (n+1)^2 - 1$

$\sum_{r=1}^{n} 2r+1 = (n+1)^2 - 1$

$2\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1 = (n+1)^2 - 1$

$\sum_{r=1}^{n} r = \frac{1}{2}\left[(n+1)^2 - 1 - n\right]$

$= \frac{1}{2}\left[n^2 + 2n + \cancel{1} - \cancel{1} - n\right]$

$= \frac{1}{2}(n^2 + n)$

$\boxed{\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)}$

(b) Given that $\sum_{r=1}^{n}(r+a) = n$, find a in terms of n. [3]

$\sum_{r=1}^{n} r + \sum_{r=1}^{n} a = n$

$\frac{1}{2}n(n+1) + an = n$

$an = n - \frac{1}{2}n(n+1)$

$a = 1 - \frac{1}{2}(n+1)$

$a = 1 - \frac{1}{2}n - \frac{1}{2}$

$\boxed{a = \frac{1}{2}(1-n)}$

2 Prove by mathematical induction that, for all positive integers n,

$$1+2x+3x^2+\ldots+nx^{n-1}=\frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}.$$ [6]

for $n=1$

$1x^{1-1} = 1$ [LHS]

$\frac{1-(1+1)x^1+1x^{1+1}}{(1-x)^2} = \frac{1-2x+x^2}{(1-x)^2} = \frac{(1-x)^2}{(1-x)^2} = 1$ (RHS)

LHS = RHS

Suppose true for $n=k$

$1+2x+\ldots kx^{k-1} = \frac{1-(k+1)x^k+kx^{k+1}}{(1-x)^2}$

Prove for $n=k+1$

$1+2x+3x^2 --- + kx^{k-1} + (k+1)x^k$

$\frac{1-(k+1)x^k+kx^{k+1}+(k+1)x^k(1-x)^2}{(1-x)^2}$

$\frac{1-kx^{k+1}+(k+1)x^k(-2x+x^2)}{(1-x)^2}$

$= \frac{1-(k+2)x^{k+1}+(k+1)x^{k+2}}{(1-x)^2}$

Hence induction is complete.

3 The quartic equation $x^4 + bx^3 + cx^2 + dx - 2 = 0$ has roots $\alpha, \beta, \gamma, \delta$. It is given that

$$\alpha+\beta+\gamma+\delta = 3, \qquad \alpha^2+\beta^2+\gamma^2+\delta^2 = 5, \qquad \alpha^{-1}+\beta^{-1}+\gamma^{-1}+\delta^{-1} = 6.$$

(a) Find the values of b, c and d. [6]

$\boxed{b = -(\alpha+\beta+\gamma+\delta) = -3}$

$\alpha^2+\beta^2+\gamma^2+\delta^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$

$5 = (-3)^2 - 2\Sigma\alpha\beta$

$2\Sigma\alpha\beta = 4$

$\Sigma\alpha\beta = 2$

$\boxed{c = 2}$

$\alpha^{-1}+\beta^{-1}+\gamma^{-1}+\delta^{-1} = \dfrac{\Sigma\alpha\beta\gamma}{\Sigma\alpha\beta\gamma\delta}$

$6 = \dfrac{\Sigma\alpha\beta\gamma}{-2}$

$\Sigma\alpha\beta\gamma = -12$

$\boxed{d = 12}$

(b) Given also that $\alpha^3+\beta^3+\gamma^3+\delta^3 = -27$, find the value of $\alpha^4+\beta^4+\gamma^4+\delta^4$. [2]

$\alpha^4+\beta^4+\gamma^4+\delta^4 = 3S_3 - 2S_2 - 12S_1 + 8$

$= 3(-27) - 2(5) - 12(3) + 8$

$\boxed{\alpha^4+\beta^4+\gamma^4+\delta^4 = -119}$

4 The lines l_1 and l_2 have equations

$$\mathbf{r} = -2\mathbf{i} - 3\mathbf{j} - 5\mathbf{k} + \lambda(-4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) \qquad \text{and} \qquad \mathbf{r} = 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \mu(2\mathbf{i} - 3\mathbf{j} + \mathbf{k})$$

respectively.

(a) Find the shortest distance between l_1 and l_2. [5]

$$\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ -3 \\ -5 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 8 \end{pmatrix}$$

$$\vec{n} = \begin{vmatrix} i & j & k \\ -4 & 3 & 5 \\ 2 & -3 & 1 \end{vmatrix} = i\begin{vmatrix} 3 & 5 \\ -3 & 1 \end{vmatrix} - j\begin{vmatrix} -4 & 5 \\ 2 & 1 \end{vmatrix} + k\begin{vmatrix} -4 & 3 \\ 2 & -3 \end{vmatrix}$$

$$= \begin{pmatrix} 18 \\ 14 \\ 6 \end{pmatrix} \approx \begin{pmatrix} 9 \\ 7 \\ 3 \end{pmatrix}$$

$$\Rightarrow \frac{1}{\sqrt{9^2+7^2+3^2}} \left| \begin{bmatrix} 9 \\ 7 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} \right| = \frac{1}{\sqrt{139}} \times 67$$

$$\boxed{= 5.68}$$

The plane Π contains l_1 and the point with position vector $-\mathbf{i}-3\mathbf{j}-4\mathbf{k}$.

(b) Find an equation of Π, giving your answer in the form $ax+by+cz=d$. [4]

$$\vec{n} = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ -4 & 3 & 5 \end{vmatrix} = i\begin{vmatrix} 0 & 1 \\ 3 & 5 \end{vmatrix} - j\begin{vmatrix} 1 & 1 \\ -4 & 5 \end{vmatrix} + k\begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix}$$

$$= \begin{bmatrix} 3 \\ 9 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

$x+3y-z=d$

$(-1)+3(-3)-(-4) = -6$

$\boxed{x+3y-z=-6}$

5 Let k be a constant. The matrices **A**, **B** and **C** are given by

$$\mathbf{A} = \begin{pmatrix} 1 & k & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 5 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 & -2 \\ -1 & 3 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{C} = \begin{pmatrix} -2 & -1 & 1 \\ 1 & 1 & 3 \end{pmatrix}.$$

It is given that **A** is singular.

(a) Show that $\mathbf{CAB} = \begin{pmatrix} 3 & -7 \\ -9 & 3 \end{pmatrix}$. [5]

$\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} - k\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 0 \Rightarrow -1-k+3=0 \Rightarrow k=-1+3$

$k=2$

$AB = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 5 \end{pmatrix}\begin{pmatrix} 0 & -2 \\ -1 & 3 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -1 & -1 \\ -2 & 0 \end{pmatrix}$

$\begin{pmatrix} -2 & -1 & 1 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ -1 & -1 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 3 & -7 \\ -9 & 3 \end{pmatrix}$

Hence shown.

(b) Find the equations of the invariant lines, through the origin, of the transformation in the x−y plane represented by **CAB**. [5]

$\begin{pmatrix} 3 & -7 \\ -9 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x-7y \\ -9x+3y \end{pmatrix} = \begin{pmatrix} x \\ mx \end{pmatrix}$

$-9x+3mx = m(3x-7mx)$

$-9+\cancel{3m} = \cancel{3m}-7m^2$

$7m^2=9$

$m^2=\frac{9}{7}$

$m=\pm\frac{3}{\sqrt{7}}$

$\boxed{y=\frac{3}{\sqrt{7}}x, \quad y=-\frac{3}{\sqrt{7}}x}$

(c) The matrices **D**, **E** and **F** represent geometrical transformations in the $x-y$ plane.

- **D** represents an enlargement, centre the origin.
- **E** represents a stretch parallel to the x-axis.
- **F** represents a reflection in the line $y = x$.

Given that **CAB** = **D** − 9**EF**, find **D**, **E** and **F**. [5]

$D = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$, $E = \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\begin{pmatrix} 3 & -7 \\ -9 & 3 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} - 9\begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix}$

$\alpha = 3$, $\beta = \frac{7}{9}$

$\Rightarrow$ $\boxed{D = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \quad E = \begin{pmatrix} 7/9 & 0 \\ 0 & 1 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$

6 **(a)** Show that the curve with Cartesian equation

$$\left(x-\tfrac{1}{2}\right)^2+y^2=\tfrac{1}{4}$$

has polar equation $r=\cos\theta$. [3]

$x^2-x+\frac{1}{4}+y^2=\frac{1}{4} \Rightarrow r^2-r\cos\theta+\cancel{\tfrac{1}{4}}=\cancel{\tfrac{1}{4}}$

$r^2=r\cos\theta$

$\boxed{r=\cos\theta}$

The curves C_1 and C_2 have polar equations

$$r=\cos\theta \quad \text{and} \quad r=\sin 2\theta$$

respectively, where $0\leqslant\theta\leqslant\frac{1}{2}\pi$. The curves C_1 and C_2 intersect at the pole and at another point P.

(b) Find the polar coordinates of P. [3]

$\sin 2\theta=\cos\theta$

$2\sin\theta\cos\theta=\cos\theta$

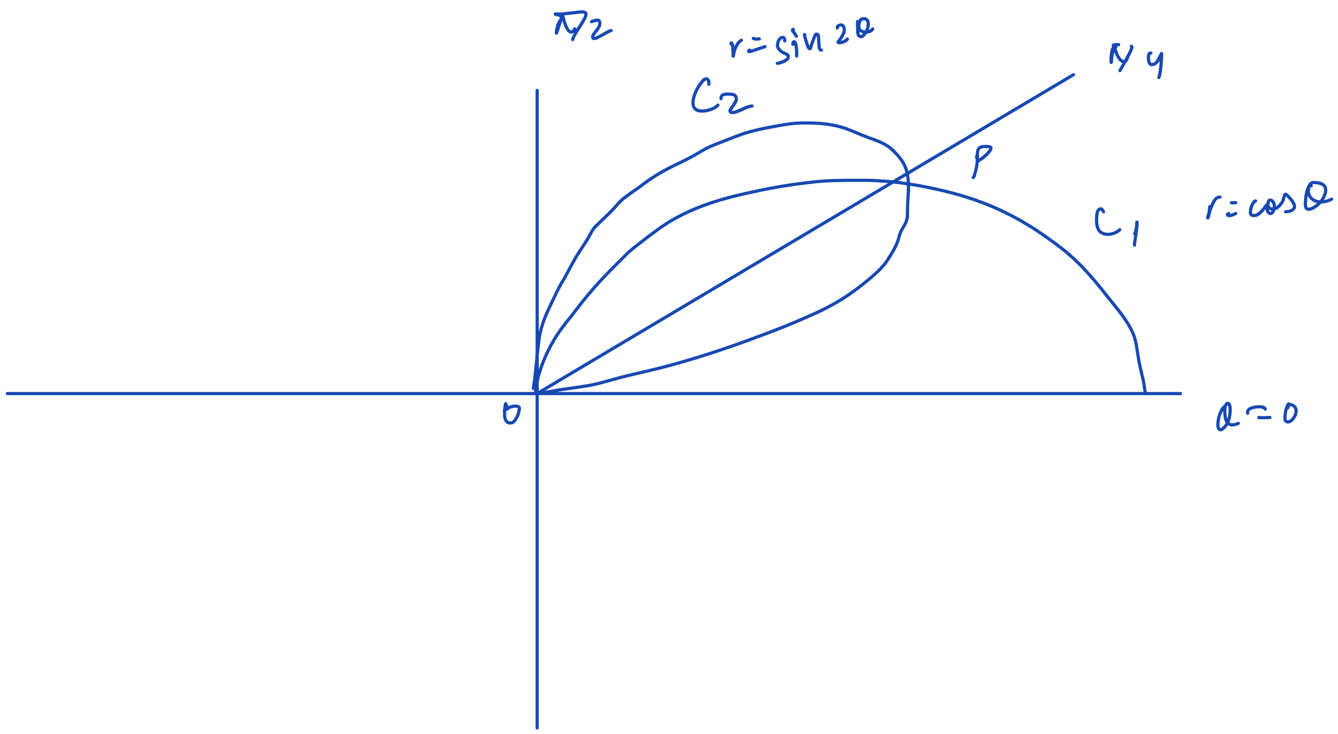
$\sin\theta=\frac{1}{2}$

$\theta=\frac{\pi}{6}$

$r=\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$

$\boxed{\left(\frac{\sqrt{3}}{2},\frac{\pi}{6}\right)}$

(c) In a single diagram sketch C_1 and C_2, clearly identifying each curve, and mark the point P. [3]



(d) The region R is enclosed by C_1 and C_2 and includes the line OP.

Find, in exact form, the area of R. [6]

$\frac{1}{2}\int_0^{\pi/6} \sin^2 2\theta \, d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2} \cos^2\theta \, d\theta$

$\frac{1}{4}\int_0^{\pi/6} 1 - \cos 4\theta \, d\theta + \frac{1}{4}\int_{\pi/6}^{\pi/2} 1 + \cos 2\theta \, d\theta$

$\frac{1}{4}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/6} + \frac{1}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\pi/6}^{\pi/2}$

$\frac{1}{4}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right) + \frac{1}{4}\left(\frac{\pi}{2} - \frac{\pi}{6} - \frac{\sqrt{2}}{4}\right) = \boxed{\frac{\pi}{8} - \frac{3\sqrt{3}}{32}}$

7 The curve C has equation $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \dfrac{x^2+2}{x^2-x-2}$.

(a) Find the equations of the asymptotes of C. [2]

$\boxed{x=-1,\ x=2 \quad y=1}$

$x^2-x-2=0$

$x^2-2x+x-2=0$

$x(x-2)+1(x-2)=0$

$(x+1)(x-2)=0$

$x=-1,\ 2.$

(b) Find the coordinates of any stationary points on C, giving your answers correct to 1 decimal place. [4]

$$\frac{dy}{dx} = \frac{(x^2-x-2)(2x)-(x^2+2)(2x-1)}{(x^2-x-2)^2} = 0$$

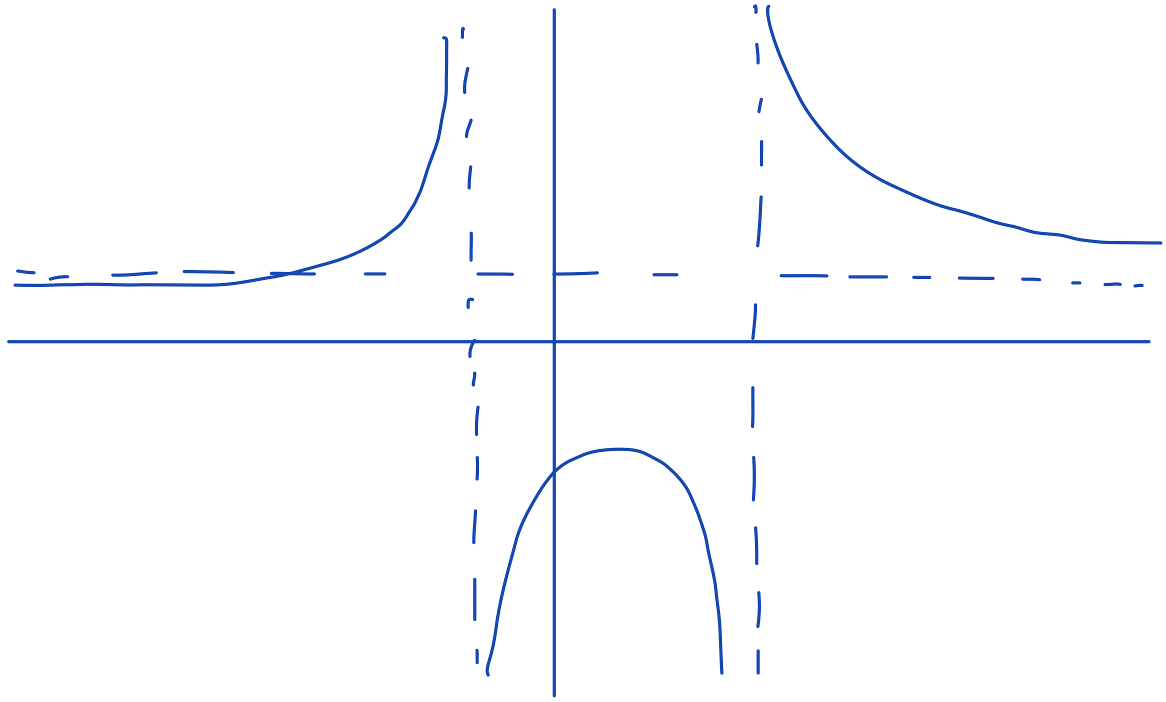
$x^2+8x-2=0$

$$x = \frac{-8 \pm \sqrt{8^2-4(1)(-2)}}{2(1)}$$

$$x = \frac{-8 \pm \sqrt{72}}{2}$$

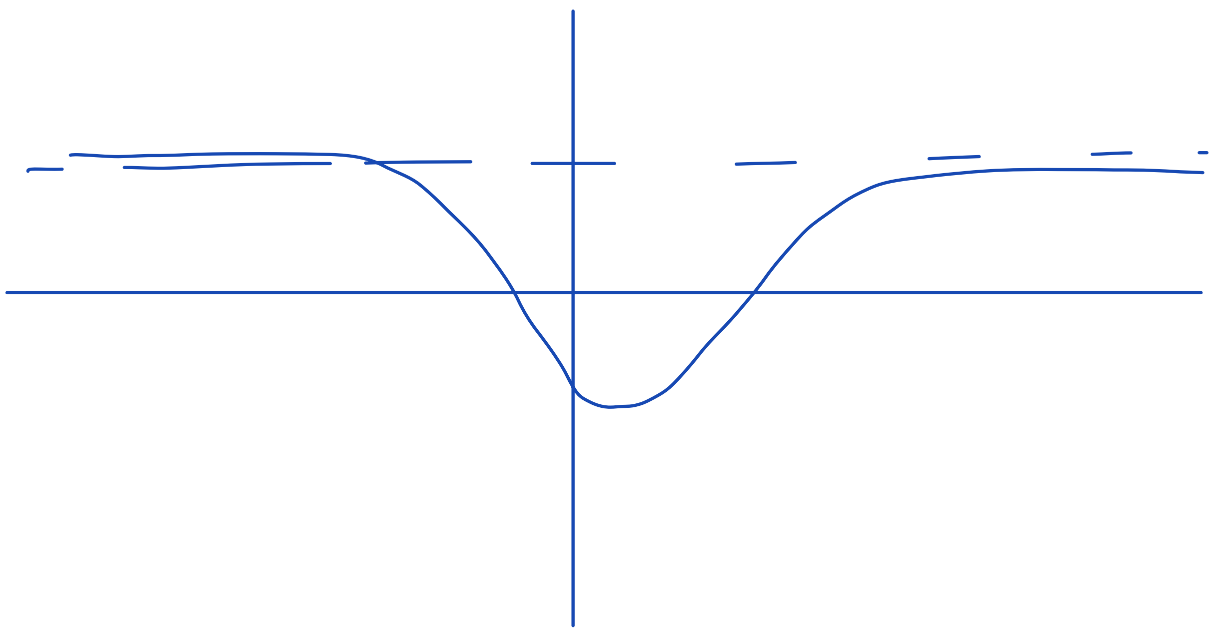
$\boxed{(-8.2,\ 0.9) \quad (0.2,\ -0.9)}$

(c) Sketch C, stating the coordinates of any intersections with the axes. [3]



(0, −1)

(d) Sketch the curve with equation $y = \dfrac{1}{f(x)}$. [2]



(e) Find the set of values for which $\frac{1}{f(x)} < f(x)$. [4]

$f^2(x) > 1$

$f(x) = 1$ $f(x) = -1$

$\frac{x^2+x}{x^2-x-2} = 1$ $\frac{x^2+x}{x^2-x-2} = -1$

$x+4=0$ $2x^2-x=0$

$\underline{x=-4}$ $x(2x-1)=0$

$\underline{x=0}$, $\underline{x=1/2}$

$\boxed{-4<x<-1 \;,\; 0<x<1/2 \;,\; x>2}$

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

