1 Let $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$.

(a) Prove by mathematical induction that, for all positive integers n,

$$2\mathbf{A}^n = \begin{pmatrix} 2 \times 3^n & 0 \\ 3^n - 1 & 2 \end{pmatrix}.$$ [5]

for $n=1$

$$2A = \begin{pmatrix} 2\times 3 & 0 \\ 3-1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 2 & 2 \end{pmatrix} = 2\begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$$

so true

Assume true for $n=k$

$$2A^k = \begin{pmatrix} 2\times 3^k & 0 \\ 3^k-1 & 2 \end{pmatrix}$$

Prove for $n=k+1$

$$2A^{k+1} = 2A^k \times A$$

$$= \begin{pmatrix} 2\times 3^k & 0 \\ 3^k-1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2\times 3^k \times 3 & 0 \\ 3(3^k-1)+2 & 2 \end{pmatrix} = \begin{pmatrix} 2\times 3^{k+1} & 0 \\ 3^{k+1}-3+2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2\times 3^{k+1} & 0 \\ 3^{k+1}-1 & 2 \end{pmatrix}$$

$\boxed{\text{Hence induction is complete.}}$

(b) Find, in terms of n, the inverse of $\mathbf{A}^n$. [2]

$$\det(A^n) = \det\begin{pmatrix} 3^n & 0 \\ \frac{1}{2}(3^n-1) & 1 \end{pmatrix} = 3^n$$

$$\boxed{A^{-n} = 3^{-n}\begin{pmatrix} 1 & 0 \\ \frac{1}{2}(1-3^n) & 3^n \end{pmatrix}}$$

2 The cubic equation $x^3 + 4x^2 + 6x + 1 = 0$ has roots α, β, γ.

(a) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [2]

$\alpha^2+\beta^2+\gamma^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$

$= (-4)^2 - 2(6)$

$\boxed{\alpha^2+\beta^2+\gamma^2 = 4}$

(b) Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{n}\left((\alpha+r)^2+(\beta+r)^2+(\gamma+r)^2\right) = n(n^2+an+b),$$

where a and b are constants to be determined. [6]

$(\alpha+r)^2+(\beta+r)^2+(\gamma+r)^2 = \alpha^2+2\alpha r+r^2+\beta^2+2\beta r+r^2+\gamma^2+2\beta r+r^2$

$= \alpha^2+\beta^2+\gamma^2+2r(\alpha+\beta+\gamma)+3r^2$

$= 4 + 2r(-4)+3r^2$

$= 3r^2-8r+4$

$3\sum_{r=1}^{n} r^2 - 8\sum_{r=1}^{n} r + \sum_{r=1}^{n} 4$

$\frac{1}{2}n(n+1)(2n+1) - 4n(n+1)+4n$

$-4n^2+\frac{1}{2}n(n+1)(2n+1)$

$= n\left(-4n+\frac{1}{2}(2n^2+3n+1)\right)$

$= n\left(n^2-\frac{5}{2}n+\frac{1}{2}\right)$

$\boxed{a=-\frac{5}{2},\ b=\frac{1}{2}}$

3 (a) Use the method of differences to find $\sum_{r=1}^{n} \frac{1}{(kr+1)(kr-k+1)}$ in terms of n and k , where k is a positive constant. [4]

$$\frac{1}{(kr+1)(kr-k+1)} = \frac{A}{kr+1} + \frac{B}{kr-k+1}$$

$$A(kr-k+1) + B(kr+1) = 1$$

$$A = -\frac{1}{k} \qquad B = \frac{1}{k}$$

$$= \frac{1}{k}\left[\frac{1}{k(r-1)+1} - \frac{1}{kr+1}\right]$$

$$\frac{1}{k}\sum_{r=1}^{n}\left[\frac{1}{k(r-1)+1} - \frac{1}{kr+1}\right]$$

$r=1$: $\frac{1}{1} - \cancel{\frac{1}{k+1}}$ $\Rightarrow$ $1 - \frac{1}{kn+1}$

$r=2$: $\cancel{\frac{1}{k+1}} - \cancel{\frac{1}{2k+1}}$

$r=n-1$: $\cancel{\frac{1}{k(n-2)+1}} - \cancel{\frac{1}{k(n-1)+1}}$

$r=n$: $\cancel{\frac{1}{k(n-1)+1}} - \frac{1}{kn+1}$

$$\boxed{\Rightarrow \frac{1}{k}\left[1 - \frac{1}{kn+1}\right]}$$

(b) Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{(kr+1)(kr-k+1)}$. [1]

$\frac{1}{k}$ since As $r \to \infty$, $\frac{1}{kn+1} \to 0$.

(c) Find also $\sum_{r=n}^{n^2} \frac{1}{(kr+1)(kr-k+1)}$ in terms of n and k. [2]

$\sum_{r=n}^{n^2} \frac{1}{(kr+1)(kr-k+1)} = \sum_{r=1}^{n^2} \frac{1}{(kr+1)(kr-k+1)} - \sum_{r=1}^{n-1} \frac{1}{(kr+1)(kr-k+1)}$

$\frac{1}{k}\left(1 - \frac{1}{kn^2+1} - \left(1 - \frac{1}{kn-k+1}\right)\right)$

$= \frac{1}{k}\left(\cancel{1} - \frac{1}{kn^2+1} - \cancel{1} + \frac{1}{k(n-1)+1}\right)$

$\boxed{= \frac{1}{k}\left(\frac{1}{k(n-1)+1} - \frac{1}{kn^2+1}\right)}$

4 The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} a & b^2 \\ c^2 & a \end{pmatrix}$, where a, b, c are real constants and $b \neq 0$.

(a) Show that $\mathbf{M}$ does not represent a rotation about the origin. [2]

$\begin{pmatrix} a & b^2 \\ c^2 & a \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$b^2 = -c^2$ Impossible since b & c are real & $b \neq 0$

(b) Find the equations of the invariant lines, through the origin, of the transformation in the x–y plane represented by $\mathbf{M}$. [5]

$\begin{pmatrix} a & b^2 \\ c^2 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + b^2y \\ c^2x + ay \end{pmatrix}$

$c^2x + ay = m(ax + b^2y)$

$c^2x + amx = m(ax + b^2mx)$

$c^2 + am = m(a + b^2m)$

$c^2 + \cancel{am} = \cancel{a}m + b^2m^2$

$m^2b^2 = c^2$

$m = \pm\sqrt{\dfrac{c^2}{b^2}}$

$\boxed{y = \dfrac{c}{b}x \;,\; y = -\dfrac{c}{b}x}$

It is given that **M** represents the sequence of two transformations in the $x-y$ plane given by an enlargement, centre the origin, scale factor 5 followed by a shear, x-axis fixed, with $(0,1)$ mapped to $(5,1)$.

(c) Find **M**. [3]

$M = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 5 & 25 \\ 0 & 5 \end{pmatrix}$

(d) The triangle DEF in the $x-y$ plane is transformed by **M** onto triangle PQR.

Given that the area of triangle DEF is 12 cm^2, find the area of triangle PQR. [2]

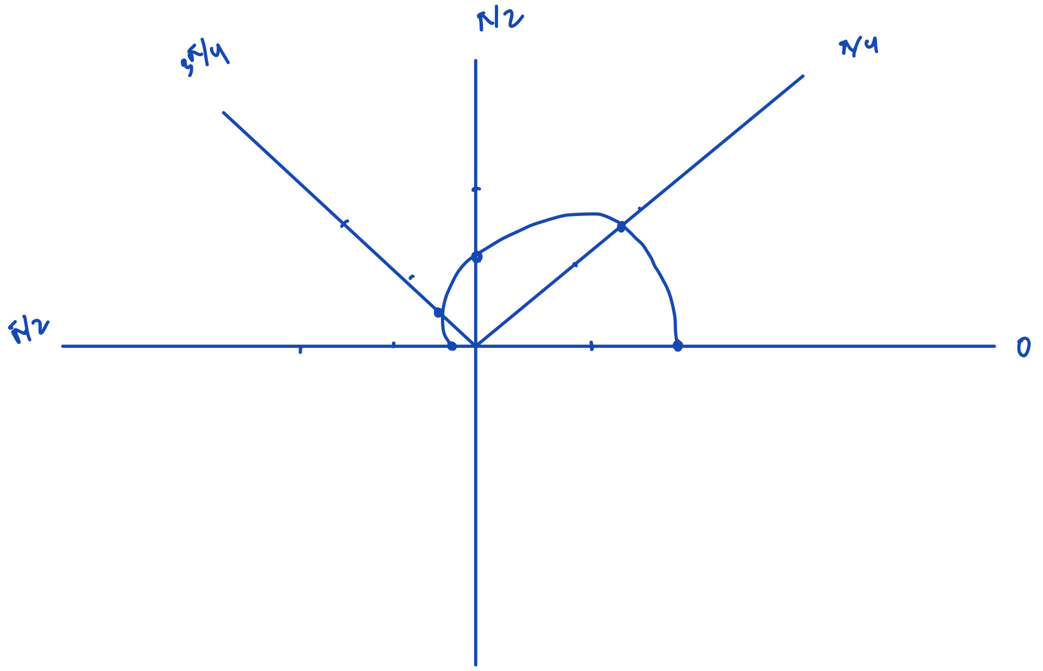
$12 \times \det(M)$

$12 \times (25)$

$\boxed{300\text{ cm}^2}$

5 The curve C has polar equation $r^2 = \frac{1}{\theta^2 + 1}$, for $0 \leqslant \theta \leqslant \pi$.

(a) Sketch C and state the polar coordinates of the point of C furthest from the pole. [3]



(1, 0)

(b) Find the area of the region enclosed by C, the initial line, and the half-line $\theta = \pi$. [4]

$\frac{1}{2}\int_0^{\pi} \frac{1}{\theta^2+1}\, d\theta$

$= \frac{1}{2}\left[\tan^{-1}\theta\right]_0^{\pi}$

$= \frac{1}{2}\left[\tan^{-1}\pi - \tan^{-1}0\right]$

$A = \boxed{\frac{1}{2}\tan^{-1}\pi = 0.631}$

(c) Show that, at the point of C furthest from the initial line,

$$\left(\theta + \frac{1}{\theta}\right)\cot\theta - 1 = 0$$

and verify that this equation has a root between 1.1 and 1.2. [5]

$$y = \frac{\sin\theta}{\sqrt{\theta^2+1}}$$

$$\frac{dy}{d\theta} = \frac{\sqrt{\theta^2+1}\cos\theta - \frac{\theta}{\sqrt{\theta^2+1}}\sin\theta}{\theta^2+1} = 0$$

$$(\theta^2+1)^{1/2}\cos\theta - \frac{\theta}{\sqrt{\theta^2+1}}\sin\theta = 0$$

$$(\theta^2+1)^{1/2}\cot\theta - \frac{\theta}{\sqrt{\theta^2+1}} = 0$$

$$(\theta^2+1)\cot\theta - \theta = 0$$

$$\theta\left(\theta + \frac{1}{\theta}\right)\cot\theta - 1 = 0$$

$$\left(\theta + \frac{1}{\theta}\right)\cot\theta - 1 = 0$$

$$\left(1.1 + \frac{1}{1.1}\right)\cot 1.1 - 1 = 0.02$$

$$\left(1.2 - \frac{1}{1.2}\right)\cot 1.2 - 1 = -0.209$$

sign change

6 The curve C has equation $y = \dfrac{x^2 + 2x - 15}{x - 2}$.

(a) Find the equations of the asymptotes of C. [3]

$x - 2 = 0$

$\boxed{x = 2}$

$x^2 + 2x - 15 = (x-2)(x+4) - 7$

$\Rightarrow \boxed{y = x - 4}$

(b) Show that C has no stationary points. [3]

$\dfrac{dy}{dx} = \dfrac{(x-2)(2x+2) - (x^2 + 2x - 15)}{(x-2)^2}$

$x^2 - 4x + 11 = 0$

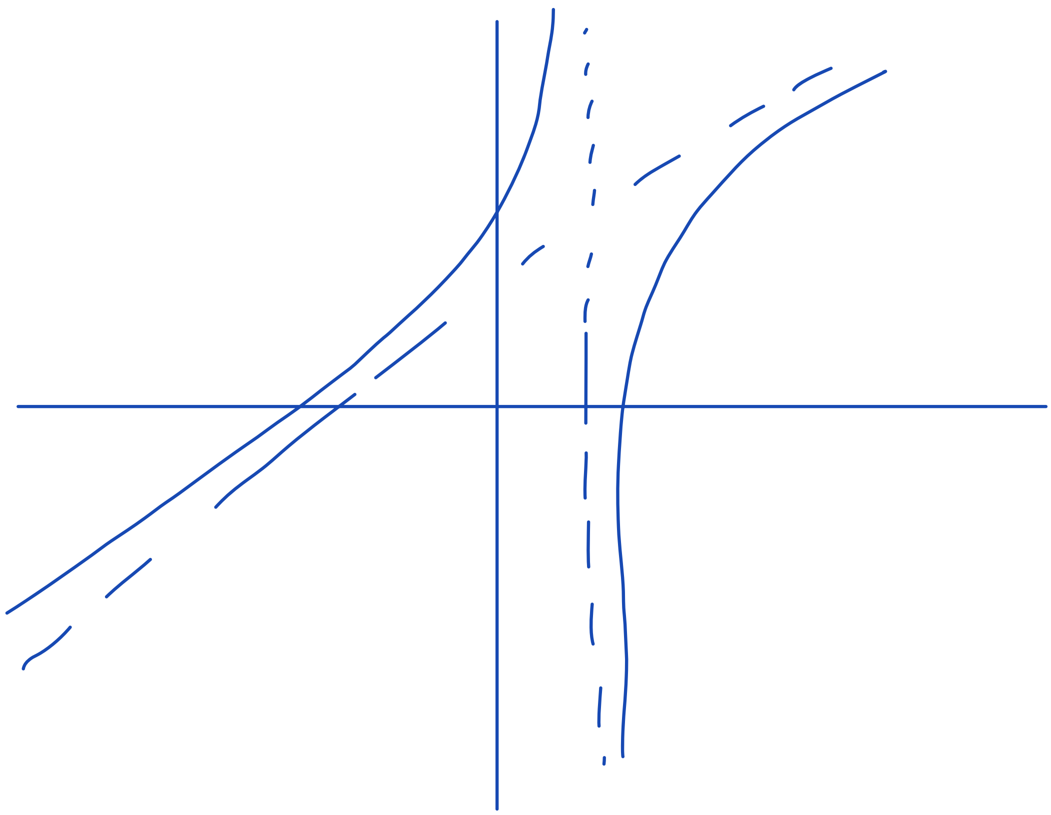
$b^2 - 4ac$

$16 - 44$

$-28 < 0$

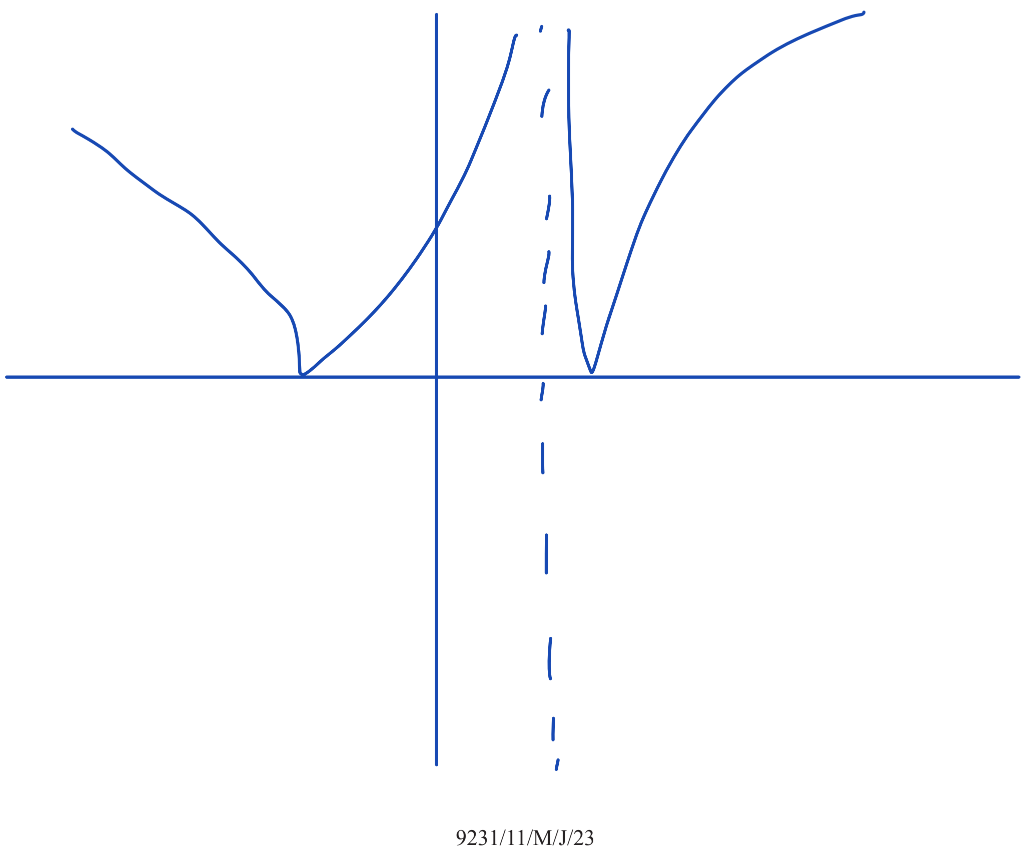
so no sols $\Rightarrow$ no t.ps.

(c) Sketch C, stating the coordinates of the intersections with the axes. [3]



(0, 7·5), (-5, 0), (3, 0)

(d) Sketch the curve with equation $y = \left|\dfrac{x^2 - 2x - 15}{x - 2}\right|$. [2]



(e) Find the set of values of x for which $\left|\frac{2x^2+4x-30}{x-2}\right| < 15$. [4]

$\frac{x^2+2x-15}{x-2} = \frac{15}{2}$

$x^2 - \frac{11}{2}x = 0$

$x(x - \frac{11}{2}) = 0$

$x = 0, \frac{11}{2}$

$\frac{x^2+2x-15}{x-2} = \frac{15}{2}$

$x^2 + \frac{19}{2}x - 30 = 0$

$x = -12, \frac{5}{2}$

$\boxed{-12 < x < 0 \qquad \frac{5}{2} < x < \frac{11}{2}}$

7 The plane Π_1 has equation $r = -4\mathbf{j} - 3\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k}) + \mu(\mathbf{i} + \mathbf{j} - \mathbf{k})$.

(a) Obtain an equation of Π_1 in the form $px + qy + rz = d$. [4]

$\vec{n} = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = i\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} - j\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + k\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$

$= \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$y + z = d$

$(-4) + (-3) = -7$

$\boxed{y + z = -7}$

(b) The plane Π_2 has equation $\mathbf{r}.(-5\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) = 4$.

Find a vector equation of the line of intersection of Π_1 and Π_2. [4]

common point $= \begin{pmatrix} -7 \\ -2 \\ -5 \end{pmatrix}$

$\vec{n} = \begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ -5 & 3 & 5 \end{vmatrix} = i\begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix} - j\begin{vmatrix} 0 & 1 \\ -5 & 5 \end{vmatrix} + k\begin{vmatrix} 0 & 1 \\ -5 & 3 \end{vmatrix}$

$= \begin{pmatrix} 2 \\ -5 \\ 5 \end{pmatrix}$

$\boxed{r = \begin{pmatrix} -7 \\ -2 \\ -5 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -5 \\ 5 \end{pmatrix}}$

The line l passes through the point A with position vector $a\mathbf{i}+a\mathbf{j}+(a-7)\mathbf{k}$ and is parallel to $(1-b)\mathbf{i}+b\mathbf{j}+b\mathbf{k}$, where a and b are positive constants.

(c) Given that the perpendicular distance from A to Π_1 is $\sqrt{2}$, find the value of a. [2]

$$\left|\frac{1}{\sqrt{2}}\left(a+a-(7-7)\right)\right| = \sqrt{2}$$

$$a+a=2$$

$$2a=2$$

$$\boxed{a=1}$$

(d) Given that the obtuse angle between l and Π_1 is $\frac{3}{4}\pi$, find the exact value of b. [4]

$$\left|\frac{b+b}{\sqrt{2}\sqrt{(1-b)^2+2b^2}}\right| = \frac{\sqrt{2}}{2}$$

$$2b=\sqrt{(1-b)^2+2b^2}$$

$$4b^2=1-2b+b^2+2b^2$$

$$3b^2-2b+1=4b^2$$

$$b^2+2b-1=0$$

$\boxed{b=-1+\sqrt{2}}$ ← since b is +ve.

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.