1 The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} k & 1 & 0 \\ 6 & 5 & 2 \\ -1 & 3 & -k \end{pmatrix},$$

where k is a real constant.

(a) Show that **A** is non-singular. [3]

$\det(A) = k\begin{vmatrix} 5 & 2 \\ 3 & -k \end{vmatrix} - 1\begin{vmatrix} 6 & 2 \\ -1 & -k \end{vmatrix} + 0\begin{vmatrix} 6 & 5 \\ -1 & 3 \end{vmatrix}$

$= k(-5k-6) - 1(-6k+2) + 0$

$= -5k^2 - \cancel{6k} + \cancel{6k} - 2$

$\Rightarrow -5k^2 - 2$

$-5k^2 - 2 = 0$

$5k^2 + 2 = 0$

$b^2 - 4ac$

$0^2 - 4(5)(2)$

$\Rightarrow -40 < 0$ so no solution

so non-singular.

(b) Given that $\mathbf{A}^{-1} = \begin{pmatrix} 3 & 0 & -1 \\ 1 & 0 & 0 \\ -\frac{23}{2} & \frac{1}{2} & 3 \end{pmatrix}$, find the value of k. $\mathbf{A} = \begin{pmatrix} k & 1 & 0 \\ 6 & 5 & 2 \\ -1 & 3 & -k \end{pmatrix},$ [2]

$\begin{vmatrix} 5 & 2 \\ 3 & -k \end{vmatrix} \Rightarrow -5k - 6 = 5k+6$

$\det(A) = -5k^2 - 2 = 5k^2 + 2$

$\dfrac{5k+6}{5k^2+2} = 3$

$15k^2 + \cancel{6} - 5k - \cancel{6} = 0$

$15k^2 - 5k = 0$

$3k^2 - k = 0$

$k(3k-1) = 0$

$k = 0 \qquad k = \frac{1}{3}$ X

$AA^{-1} = I$

$\begin{pmatrix} k & 1 & 0 \\ 6 & 5 & 2 \\ -1 & 3 & -k \end{pmatrix}\begin{pmatrix} 3 & 0 & -1 \\ 1 & 0 & 0 \\ -\frac{23}{2} & \frac{1}{2} & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$3k + 1 = 1$

$3k = 0$

$k = 0$

$\boxed{\text{so } k = 0}$

2 The cubic equation $x^3+2x^2+3x+1=0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^2+1, \beta^2+1, \gamma^2+1$. [3]

$y = x^2+1$

$x^2 = y-1$

$(x^3+3x)^2 = (2x^2+1)^2$

$x^6+6x^4+9x^2 = 4x^4+4x^2+1$

$x^6+2x^4+5x^2-1=0$

$(x^2)^3+2(x^2)^2+5(x^2)-1=0$

$(y-1)^3+2(y-1)^2+5(y-1)-1=0$

$y^3-3y^2+3y-1+2(y^2-2y+1)+5y-6=0$

$y^3-3y^2+3y-1+2y^2-4y+2+5y-6=0$

$y^3-y^2+4y-5=0$

$\boxed{y^3-y^2+4y-5=0}$

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(b) Find the value of $(\alpha^2+1)^2+(\beta^2+1)^2+(\gamma^2+1)^2$. [2]

$(\alpha^2+1)^2+(\beta^2+1)^2+(\gamma^2+1)^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$

$= (1)^2 - 2(4)$

$= -7$

$\boxed{(\alpha^2+1)^2+(\beta^2+1)^2+(\gamma^2+1)^2 = -7}$

(c) Find the value of $(\alpha^2+1)^3+(\beta^2+1)^3+(\gamma^2+1)^3$. [2]

$y^3 - y^2 + 4y - 5 = 0$

$S_3 - S_2 + 4S_1 - 15 = 0$

$(\alpha^2+1)^3 - (\alpha^2+1)^2 + 4(\alpha^2+1) - 5 = 0$

$+ (\beta^2+1)^3 - (\beta^2+1)^2 + 4(\beta^2+1) - 5 = 0$

$+ (\gamma^2+1)^3 - (\gamma^2+1)^2 + 4(\gamma^2+1) - 5 = 0$

$S_3 = S_2 - 4S_1 + 15$

$= -7 - 4 + 15$

$= 4$

$\boxed{(\alpha^2+1)^3+(\beta^2+1)^3+(\gamma^2+1)^3 = 4}$

3 The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix}$.

(a) The matrix $\mathbf{M}$ represents a sequence of two geometrical transformations in the x-y plane.

Give full details of each transformation, and make clear the order in which they are applied. [4]

Stretch parallel to x-axis scale factor 7 followed by shear (x-axis fixed) $\rightarrow$ (0,1) mapped to (2,1)

(b) Find the equations of the invariant lines, through the origin, of the transformation represented by $\mathbf{M}$. [5]

$M = \begin{pmatrix} 7 & 2 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 7 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$

$7x + 2y = X \qquad y = Y$

$7x + 2mx = X \qquad y = mX$

$\dfrac{y}{7x + 2mx} = m$

$7mx + 2m^2x = mx$

$2m^2x + 6mx = 0$

$2m^2 + 6m = 0$

$m^2 + 3m = 0$

$m(m+3) = 0$

$m = 0 \qquad m = -3$

$\boxed{y = 0 \qquad y = -3x}$

The triangle *DEF* in the *x*-*y* plane is transformed by **M** onto triangle *PQR*.

(c) Given that the area of triangle *PQR* is 35 cm^2, find the area of triangle *DEF*. [2]

$\text{Det}(M) = 7 - 0 = 7$

$|DEF| \times 7 = 35$

$\boxed{|DEF| = 5\text{ cm}^2}$

4 (a) Prove by mathematical induction that, for all positive integers n,

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1).$$ [5]

for $n=1$

$\sum_{r=1}^{1} r^2 = 1^2 = 1$ (LHS)

$\frac{1}{6}(1)(2)(3) = \frac{1}{6}(6) = 1$ (RHS)

LHS = RHS

Suppose true for $n=k$

$\sum_{r=1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1)$

Prove true for $n=k+1$

$\sum_{r=1}^{k+1} r^2 = \frac{1}{6}(k+1)(k+2)(2k+3)$

$\sum_{r=1}^{k+1} r^2 = \sum_{r=1}^{k} r^2 + (k+1)^2$

$= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$

$= \frac{1}{6}(k+1)[k(2k+1) + 6(k+1)]$

$= \frac{1}{6}(k+1)[2k^2 + k + 6k + 6]$

$= \frac{1}{6}(k+1)[2k^2 + 7k + 6]$

$= \frac{1}{6}(k+1)[2k^2 + 4k + 3k + 6]$

$= \frac{1}{6}(k+1)[2k(k+2) + 3(k+2)]$

$= \frac{1}{6}(k+1)(k+2)(2k+3)$

$= \frac{1}{6}(k+1)(k+2) + (2(k+1)+1)$

Hence induction is complete.

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The sum S_n is defined by $S_n = \sum_{r=1}^{n} r^4$.

(b) Using the identity

$$(2r+1)^5 - (2r-1)^5 \equiv 160r^4 + 80r^2 + 2,$$

show that $S_n = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$. [6]

$\sum_{r=1}^{n} (2r+1)^5 - (2r-1)^5$

$r=1: \quad 3^5 - 1^5$

$r=2: \quad 5^5 - 3^5$

$r=3: \quad 7^5 - 5^5$

$r=n-1: \quad (2n-1)^5 - (2n-3)^5$

$r=n: \quad (2n+1)^5 - (2n-1)^5$

$\sum_{r=1}^{n} (2r+1)^5 - (2r-1)^5 = (2n+1)^5 - 1$

$(2n+1)^5 - 1 = 160\sum_{r=1}^{n} r^4 + 80\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} 2$

$160\sum_{r=1}^{n} r^4 = (2n+1)^5 - \frac{40}{3}n(n+1)(2n+1) - (2n+1)$

$= (2n+1)\left[(2n+1)^4 - \frac{40}{3}n^2 - \frac{40}{3}n - 1\right]$

$= (2n+1)\left[16n^4 + 32n^3 + 24n^2 + 8n - \frac{40}{3}n^2 - \frac{40}{3}n\right]$

$= (2n+1)\left[16n^4 + 32n^3 + \frac{32}{3}n^2 - \frac{16}{3}n\right]$

$= \frac{16}{3}n(2n+1)\left[3n^3 + 6n^2 + 2n - 1\right]$

$160S_n = \frac{16}{3}n(2n+1)(n+1)(3n^2+3n-1)$

$\boxed{S_n = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n+1)}$

(c) Find the value of $\lim_{n\to\infty}\left(n^{-5}S_n\right)$. [2]

$\lim_{n\to\infty}\left[\frac{(n+1)(2n+1)(3n^2+3n+1)}{30n^4}\right]$

$\Rightarrow \frac{6n^4}{30n^4} \Rightarrow \boxed{\frac{1}{5}}$

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5 The lines l_1 and l_2 have equations $\mathbf{r} = \mathbf{i} + 4\mathbf{j} - \mathbf{k} + \lambda(\mathbf{j} - 2\mathbf{k})$ and $\mathbf{r} = -3\mathbf{i} + 4\mathbf{j} + \mu(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ respectively.

(a) Find the shortest distance between l_1 and l_2. [5]

$a_2 - a_1 = \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$

$b_1 \times b_2 \Rightarrow \hat{n} = \begin{vmatrix} i & j & k \\ 0 & 1 & -2 \\ 1 & 2 & 1 \end{vmatrix} = i\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} - j\begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix} + k\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}$

$\Rightarrow \begin{bmatrix} 5 \\ -2 \\ -1 \end{bmatrix}$

$\sqrt{5^2 + (-2)^2 + (-1)^2} \Rightarrow \sqrt{30}$

$\Rightarrow \frac{1}{\sqrt{30}} \left|\left| \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix} \right|\right| = \frac{21}{\sqrt{30}} = \boxed{3.83}$

The plane Π_1 contains l_1 and is parallel to l_2.

(b) Obtain an equation of Π_1 in the form $px + qy + rz = s$. [2]

$\hat{n} = \begin{bmatrix} 5 \\ -2 \\ -1 \end{bmatrix}$ $(1, 4, -1)$

$5x - 2y - z = s$

$5(1) - 2(4) - (-1) = -2$

$\boxed{5x - 2y - z = -2}$

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(c) The point $(1, 1, 1)$ lies on the plane Π_2. A

B

It is given that the line of intersection of the planes Π_1 and Π_2 passes through the point $(0, 0, 2)$ and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$.

Obtain an equation of Π_2 in the form $ax + by + cz = d$. [3]

$\underset{\sim}{r}_1 = 2k + \lambda(i + 4j - 3k) = \overrightarrow{OB}$

$\underset{\sim}{r}_2 = i + j + k = \overrightarrow{OA}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -i - j + k$

$\underset{\sim}{n} = \begin{vmatrix} i & j & k \\ -1 & -1 & 1 \\ 1 & 4 & -3 \end{vmatrix} = i\begin{vmatrix} -1 & 1 \\ 4 & -3 \end{vmatrix} - j\begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix} + k\begin{vmatrix} -1 & -1 \\ 1 & 4 \end{vmatrix}$

$= \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}$

$-x - 2y - 3z = d$

$-1 - 2(1) - 3(1) = d$

$d = -1 - 2 - 3$

$d = -6$

$-x - 2y - 3z = -6$

$\boxed{x + 2y + 3z = 6}$

6 The curve C has equation $y = \dfrac{x+1}{x^2+3}$.

(a) Show that C has no vertical asymptotes and state the equation of the horizontal asymptote. [2]

$x^2 + 3 \neq 0$

$b^2 - 4ac$ $\quad \boxed{y = 0} \leftarrow \underline{HA}$

$0^2 - 4(3)$

$-12 < 0$

so no real sols.

(b) Find the coordinates of any stationary points on C. [4]

$u = x+1 \qquad v = x^2+3$

$u' = 1 \qquad v' = 2x$

$\dfrac{dy}{dx} = \dfrac{vu' - uv'}{v^2}$

$= \dfrac{(x^2+3) - 2x(x+1)}{(x^2+3)^2} = 0$

$x^2 + 3 - 2x^2 - 2x = 0$

$-x^2 - 2x + 3 = 0$

$x^2 + 2x - 3 = 0$

$x^2 + 3x - x - 3 = 0$

$(x+3)(x-1) = 0$

$x = -3 \quad x = 1$

$y = \dfrac{(-3+1)}{(-3)^2+3} \qquad y = \dfrac{1+1}{1^2+3}$

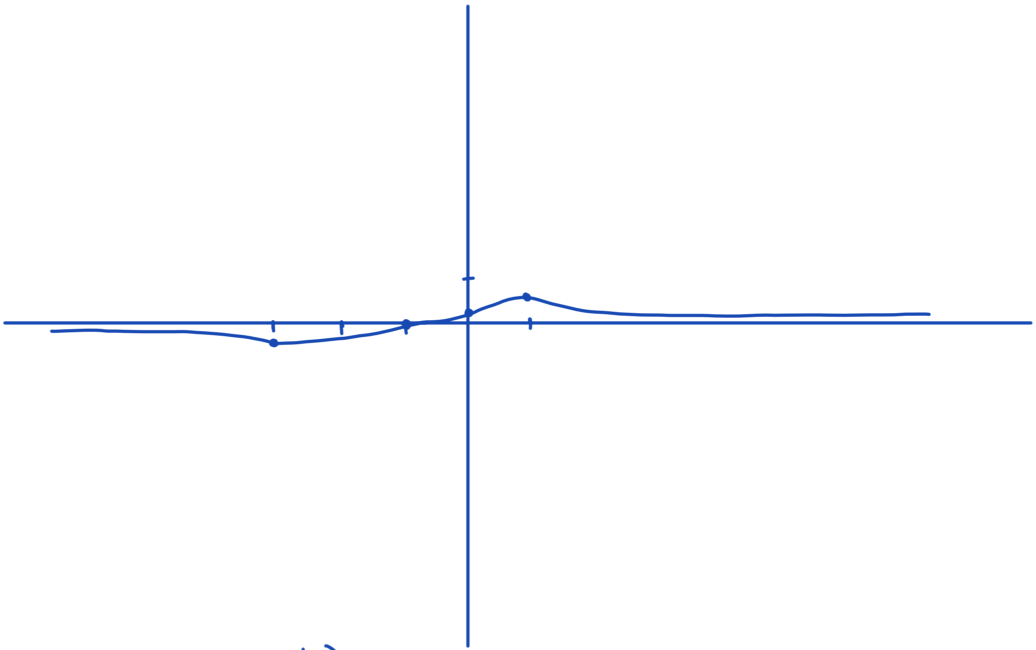
$$\boxed{\begin{array}{ll} x = -3 & x = 1 \\ y = -\frac{1}{6} & y = \frac{1}{2} \end{array}}$$

$x=0 \quad y=0$

$y=\frac{1}{3} \quad x=-1$

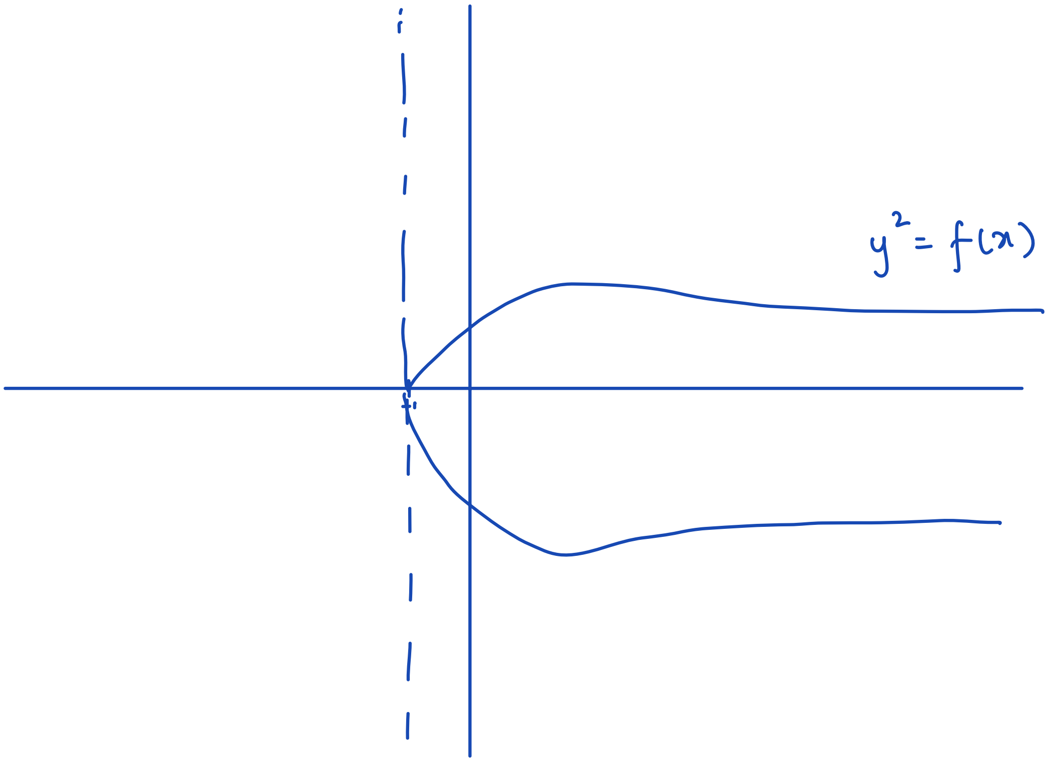
$y=\frac{x+1}{x^2+3}$

(c) Sketch C, stating the coordinates of the intersections with the axes. [3]



$(-1,0) \quad (0,\frac{1}{3})$

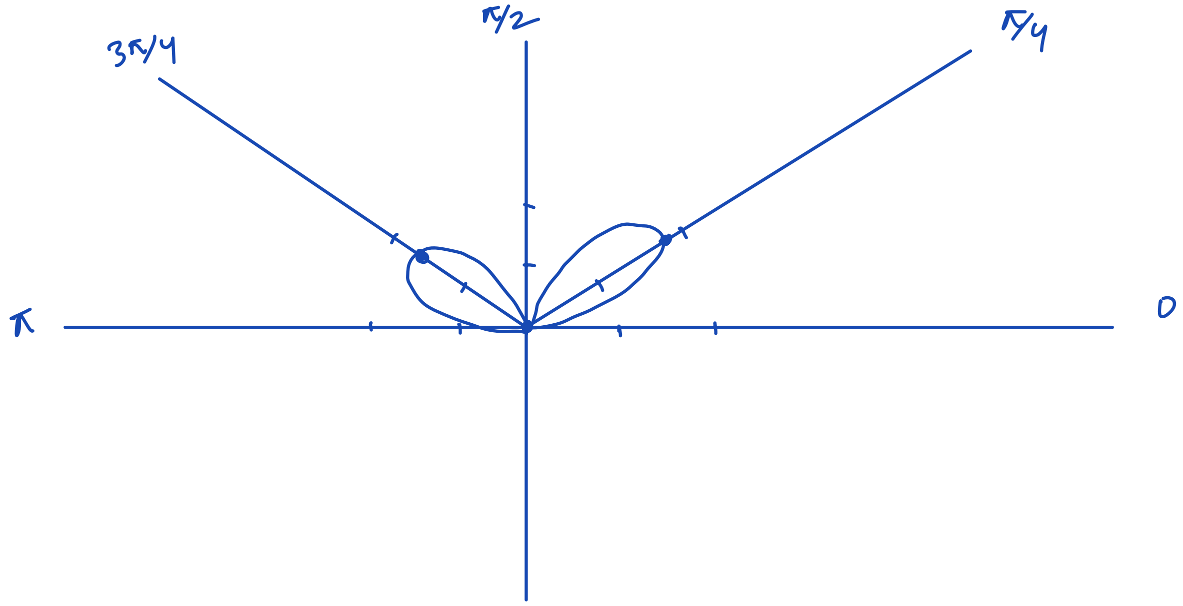
(d) Sketch $y^2 = \dfrac{x+1}{x^2+3}$, stating the coordinates of the stationary points and the intersections with the axes. [4]



$(-1,0)$ Stationary : $(0, \pm\frac{1}{\sqrt{3}})$

7 The curve C has polar equation $r^2 = \sin 2\theta \cos\theta$, for $0 \leqslant \theta \leqslant \pi$.

(a) Sketch C and state the equation of the line of symmetry. [3]



$\theta = \frac{\pi}{2}$

(b) Find a Cartesian equation for C. [3]

$r^2 = \sin 2\theta \cos\theta$ $\qquad x = r\cos\theta$

$r^2 = 2\sin\theta\cos^2\theta$ $\qquad y = r\sin\theta$

$r^2 = 2\sin\theta\cos\theta \cdot \cos\theta$

$r^5 = 2r\sin\theta \cdot r\cos\theta \cdot r\cos\theta$

$(x^2+y^2)^{5/2} = 2x^2y$

$\boxed{(x^2+y^2)^{5/2} = 2x^2y}$

(c) Find the total area enclosed by C. [4]

$r^2 = \sin 2\theta \cos\theta$

$\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}\, dx$

$A = \frac{1}{2}\int r^2\, d\theta$

$= \frac{1}{2}\int_0^{\pi} \sin 2\theta \cos\theta \; d\theta$

$\int u^n u' \, du$

$= \frac{u^{n+1}}{n+1} + C$

$n \neq -1$

$= \frac{1}{2}\int_0^{\pi} 2\sin\theta\cos^2\theta \; d\theta$

$= \int_0^{\pi} \sin\theta \cos^2\theta \; d\theta$

$= -\int_0^{\pi} (\cos\theta)^2 (-\sin\theta)\; d\theta$

$= \left[-\frac{1}{3}\cos^3\theta\right]_0^{\pi} = \boxed{\frac{2}{3}}$

(d) Find the greatest distance of a point on C from the pole. [6]

$$\frac{dr}{d\theta} = \frac{1}{2\sqrt{\sin 2\theta \cos\theta}} \cdot \left[2\cos 2\theta \cos\theta - \sin\theta \sin 2\theta\right]$$

$$2\cos 2\theta \cos\theta - \sin\theta \sin 2\theta = 6\cos^3\theta - 4\cos\theta$$

$$3\cos^3\theta - 2\cos\theta = 0$$

$$\cos\theta\left[3\cos^2\theta - 2\right] = 0$$

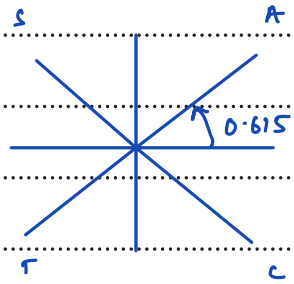
$\cos\theta = 0$

$\theta = \frac{\pi}{2}$

$\cos^2\theta = \frac{2}{3}$

$\cos\theta = \pm\sqrt{\frac{2}{3}}$

$\alpha = 0.615$



$\theta = \pi - 0.615$

$\theta = 2.526$

$r = \sqrt{\sin 2\theta \cos\theta} \Rightarrow \boxed{0.877}$