a b c d

1 The cubic equation $2x^3 + x^2 - px - 5 = 0$, where p is a positive constant, has roots α, β, γ.

(a) State, in terms of p, the value of $\alpha\beta + \beta\gamma + \gamma\alpha$. [1]

$\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a} = -\frac{p}{2}$

$\Rightarrow \boxed{-\frac{p}{2}}$

(b) Find the value of $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$. [2]

$\alpha\beta\gamma(\alpha + \beta + \gamma)$

$\left(-\frac{d}{a}\right)\left(-\frac{b}{a}\right)$

$\left(\frac{5}{2}\right)\left(-\frac{1}{2}\right)$

$\Rightarrow \boxed{-\frac{5}{4}}$

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(c) Deduce a cubic equation whose roots are $\alpha\beta, \beta\gamma, \alpha\gamma$. [1]

$4x^3 + 2px^2 - 5x - 25 = 0$

(d) Given that $\alpha^2 + \beta^2 + \gamma^2 = \frac{1}{3}$, find the value of p. [2]

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$\frac{1}{3} = \left(-\frac{1}{2}\right)^2 - 2\left(-\frac{p}{2}\right)$

$\frac{1}{3} = \frac{1}{4} + p$

$\boxed{p = \frac{1}{12}}$

2 Prove by mathematical induction that $6^{4n} + 38^n - 2$ is divisible by 74 for all positive integers n. [6]

for $k=1$

$6^4 + 38 - 2 = 1296 + 38 - 2 = 1332 = 74(18)$

Suppose true for $n=k$

$6^{4k} + 38^k - 2 = 74\ell$

Prove true for $n = k+1$

$6^{4(k+1)} + 38^{k+1} - 2$

$6^4 \cdot 6^{4k} + 38 \cdot 38^k - 2$

$1296 \cdot 6^{4k} + 38 \cdot 38^k - 2$

$1295 \cdot 6^{4k} + 37 \cdot 38^k + f(k)$ is divisible by 74

$(1295+1) \cdot 6^{4k} + (37+1) \cdot 38^k - 2$

$1295 \cdot 6^{4k} + 37 \cdot 38^k + 6^{4k} + 38^k - 2$

$37(35 \cdot 6^{4k} + 38^k) + 74\ell$

$37(2m) + 74\ell$

$74(m+\ell)$

thus divisible by 74.

Hence induction is complete.

3 **(a)** Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{N} r(r+1)(3r+4) = \tfrac{1}{12}N(N+1)(N+2)(9N+19).$$ [3]

$$\sum_{r=1}^{N} (r^2+r)(3r+4) = \sum_{r=1}^{N} (3r^3+7r^2+4r)$$

$$= 3\sum_{r=1}^{N} r^3 + 7\sum_{r=1}^{N} r^2 + 4\sum_{r=1}^{N} r$$

$$= \frac{3}{4}N^2(N+1)^2 + \frac{7}{6}N(N+1)(2N+1) + 2N(N+1)$$

$$= \frac{1}{12}N(N+1)\left[9N(N+1) + 14(2N+1) + 24\right]$$

$$= \frac{1}{12}N(N+1)\left[9N^2 + 9N + 28N + 14 + 24\right]$$

$$= \frac{1}{12}N(N+1)\left[9N^2 + 37N + 38\right]$$

$$= \frac{1}{12}N(N+1)(N+2)(9N+19).$$

(b) Express $\dfrac{3r+4}{r(r+1)}$ in partial fractions and hence use the method of differences to find

$$\sum_{r=1}^{N} \frac{3r+4}{r(r+1)}\left(\frac{1}{4}\right)^{r+1}$$

in terms of N. [4]

$$\frac{3r+4}{r(r+1)} = \frac{A}{r} + \frac{B}{r+1}$$

$$A(r+1) + Br = 3r+4$$

let $r=0$ let $r=-1$

$A = 4$ $B=-1$

$$\frac{3r+4}{r(r+1)} = \frac{4}{r} - \frac{1}{r+1}$$

$$\frac{1}{4}\sum_{r=1}^{N}\left(\frac{4}{r} - \frac{1}{r+1}\right)\left(\frac{1}{4}\right)^{r}$$

$r=1$: $\left(\frac{4}{1} - \frac{1}{2}\right)\left(\frac{1}{4}\right)^{1}$ $\Rightarrow$ $\left(1 - \cancel{\frac{1}{8}}\right)$

$r=2$: $\left(\frac{4}{2} - \frac{1}{3}\right)\left(\frac{1}{4}\right)^{2}$ $\left(\cancel{\frac{1}{8}} - \cancel{\frac{1}{48}}\right)$

$r=3$: $\left(\frac{4}{3} - \frac{1}{4}\right)\left(\frac{1}{4}\right)^{3}$ $\left(\cancel{\frac{1}{48}} - \cancel{\frac{1}{256}}\right)$

$r=4$: $\left(\frac{4}{4} - \frac{1}{5}\right)\left(\frac{1}{4}\right)^{4}$ $\vdots$

$r=5$: $\left(\frac{4}{5} - \frac{1}{6}\right)\left(\frac{1}{4}\right)^{5}$

$r=N-1$: $\left(\frac{4}{N-1} - \frac{1}{N}\right)\left(\frac{1}{4}\right)^{N-1}$ $\left(\cancel{\left(\frac{1}{4}\right)^{N-1} \cdot \frac{4}{N-1}}\right) - \left(\cancel{\left(\frac{1}{4}\right)^{N-1} \cdot \frac{1}{N}}\right)$

$r=N$: $\left(\frac{4}{N} - \frac{1}{N+1}\right)\left(\frac{1}{4}\right)^{N}$ $\left(\cancel{\left(\frac{1}{4}\right)^{N} \cdot \frac{4}{N}} - \left(\frac{1}{4}\right)^{N} \cdot \frac{1}{N+1}\right)$

$$\Rightarrow 1 - \frac{1}{(N+1)} \cdot \left(\frac{1}{4}\right)^{N}$$

$$\Rightarrow \boxed{\frac{1}{4} - \left(\frac{1}{4}\right)^{N+1} \cdot \frac{1}{N+1}}$$

(c) Deduce the value of $\displaystyle\sum_{r=1}^{\infty} \frac{3r+4}{r(r+1)}\left(\frac{1}{4}\right)^{r+1}$. [1]

$\frac{1}{4} - \left(\frac{1}{4}\right)^{N+1} \cdot \frac{1}{N+1}$ As $N \to \infty$, $\frac{1}{N+1} \to 0$

$\therefore \boxed{\frac{1}{4}}$

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$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

4 The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 14 & 0 \\ 0 & 1 \end{pmatrix}$.

(a) The matrix $\mathbf{M}$ represents a sequence of two geometrical transformations in the x-y plane.

Give full details of each transformation, and make clear the order in which they are applied. [4]

Stretch parallel to x-axis, factor 14
followed by
$\pi/3$ rotation ACW about origin.

(b) Write $\mathbf{M}^{-1}$ as the product of two matrices, neither of which is $\mathbf{I}$. [2]

$M = AB$

$M^{-1} = (AB)^{-1}$

$M^{-1} = B^{-1}A^{-1}$

$= \frac{1}{14}\begin{pmatrix} 1 & 0 \\ 0 & 14 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}$

$M^{-1} = \begin{pmatrix} \frac{1}{14} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}$

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(c) Find the equations of the invariant lines, through the origin, of the transformation represented by **M**. [5]

$M = \begin{pmatrix} 7 & -\frac{1}{2}\sqrt{3} \\ 7\sqrt{3} & \frac{1}{2} \end{pmatrix}$

$\begin{pmatrix} 7 & -\frac{1}{2}\sqrt{3} \\ 7\sqrt{3} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$7x - \frac{1}{2}\sqrt{3}y = x$

$7\sqrt{3}x + \frac{1}{2}y = y$

$y = mx$

$m = \frac{7\sqrt{3}}{3}$

$m = 2\sqrt{3}$

$\frac{7\sqrt{3}x + \frac{1}{2}y}{7x - \frac{1}{2}\sqrt{3}y} = m$

$\boxed{y = 2\sqrt{3}x \quad y = \frac{7\sqrt{3}}{3}x}$

$7\sqrt{3}x + \frac{1}{2}y = 7mx - \frac{1}{2}m\sqrt{3}y$

$7\sqrt{3}x - 7mx + \frac{1}{2}y + \frac{1}{2}m\sqrt{3}y = 0$

$7\sqrt{3}x - 7mx + \frac{1}{2}mx + \frac{1}{2}m\sqrt{3}mx = 0$

$\frac{1}{2}m^2\sqrt{3}x - \frac{13}{2}mx + 7\sqrt{3}x = 0$

$\frac{1}{2}\sqrt{3}m^2 - \frac{13}{2}m + 7\sqrt{3} = 0$

$\sqrt{3}m^2 - 13m + 14\sqrt{3} = 0$

(d) The triangle *ABC* in the *x*-*y* plane is transformed by **M** onto triangle *DEF*.

Given that the area of triangle *DEF* is 28 cm², find the area of triangle *ABC*. [2]

$\det(M) = \frac{7}{2} - \left(-\frac{7}{2}(3)\right)$

$= 14$

$14 \times |ABC| = 28$

$\boxed{|ABC| = 2\text{ cm}^2}$

5 The points A, B, C have position vectors

$$\overrightarrow{OA} \qquad \overrightarrow{OB} \qquad \overrightarrow{OC}$$

$$2\mathbf{i}+2\mathbf{j}+4\mathbf{k}, \qquad 2\mathbf{i}+4\mathbf{j}-\mathbf{k}, \qquad -3\mathbf{i}-3\mathbf{j}+4\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \cancel{2i} + 4j - k - \cancel{2i} - 2j - 4k$

$\overrightarrow{AB} = 2j - 5k$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -3i - 3j + \cancel{4k} - 2i - 2j - \cancel{4k}$

$\overrightarrow{AC} = -5i - 5j$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = -3i - 3j + 4k - 2i - 4j + k$

$\overrightarrow{BC} = -5i - 7j + 5k$

$$\underset{\sim}{n} = \begin{vmatrix} i & j & k \\ 0 & 2 & -5 \\ -5 & -5 & 0 \end{vmatrix} = -25i + 25j + 10k$$

$$= \begin{bmatrix} -5 \\ 5 \\ 2 \end{bmatrix}$$

$ax + by + cz = d$

$-5x + 5y + 2z = d$

$-5(2) + 5(2) + 2(4) = 8$

$$\boxed{-5x + 5y + 2z = 8}$$

The point D has position vector $2\mathbf{i}+\mathbf{j}+3\mathbf{k}$.

(b) Find the perpendicular distance from D to the plane ABC. [2]

$$D = \frac{|-5(2)+5(1)+2(3)-8|}{\sqrt{5^2+5^2+2^2}}$$

$$= \frac{|-7|}{\sqrt{54}} \Rightarrow \boxed{\frac{7}{\sqrt{54}} = 0.953}$$

(c) Find the shortest distance between the lines AB and CD. [5]

$$\vec{CD} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -3 \\ -3 \\ 4 \end{pmatrix} = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}$$

$$\vec{n} = \begin{vmatrix} i & j & k \\ 0 & 2 & -5 \\ 5 & 4 & -1 \end{vmatrix} = 18i - 25j - 10k = \begin{bmatrix} 18 \\ -25 \\ -10 \end{bmatrix}$$

$$\text{det} = \sqrt{18^2+25^2+10^2} = \sqrt{1049}$$

$$\frac{1}{\sqrt{1049}} \left| \begin{pmatrix} -5 \\ -5 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 18 \\ -25 \\ -10 \end{pmatrix} \right| = \boxed{\frac{35}{\sqrt{1049}} = 1.08}$$

6 The curve C has equation $y = \frac{x^2 + ax + 1}{x + 2}$, where $a > \frac{5}{2}$.

(a) Find the equations of the asymptotes of C. [3]

$x + 2 = 0$

$\boxed{x = -2}$ ← VA

$\boxed{y = x + (a-2)}$

$$\begin{array}{r|l} & x + (a-2) \\ x+2 & x^2 + ax + 1 \\ & -\ x^2 + 2x \\ \hline & (a-2)x + 1 \\ & -\ (a-2)x + 2a - 4 \\ \hline & 1 - 2a + 4 \Rightarrow 5 - 2a \end{array}$$

(b) Show that C has no stationary points. [4]

$u = x^2 + ax + 1$ $\quad v = x + 2$

$u' = 2x + a$ $\quad v' = 1$

$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$

$= \frac{(x+2)(2x+a) - (x^2+ax+1)}{(x+2)^2}$

$= 2x^2 + \cancel{ax} + 4x + 2a - x^2 - \cancel{ax} - 1$

$= \frac{x^2 + 4x + (2a-1)}{(x+2)^2}$

$\frac{dy}{dx} = 0$

$\frac{x^2 + 4x + (2a-1)}{(x+2)^2} = 0$

$x^2 + 4x + (2a-1) = 0$

$b^2 - 4ac$

$4^2 - 4(2a-1)$

$16 - 8a + 4$

$20 - 8a$

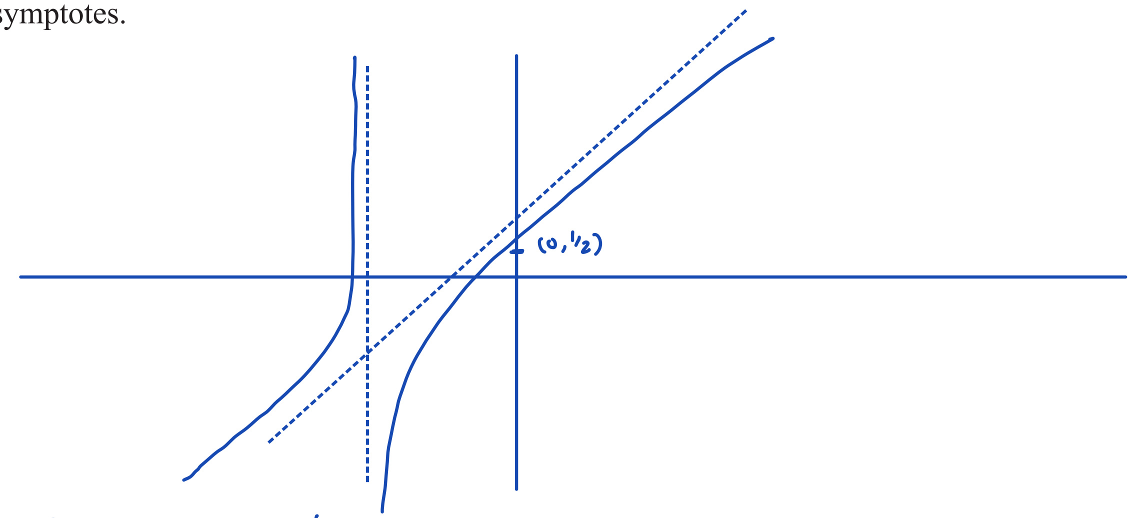
since $a > 5/2$

$20 - 8a < 0$

so NO SOLUTIONS

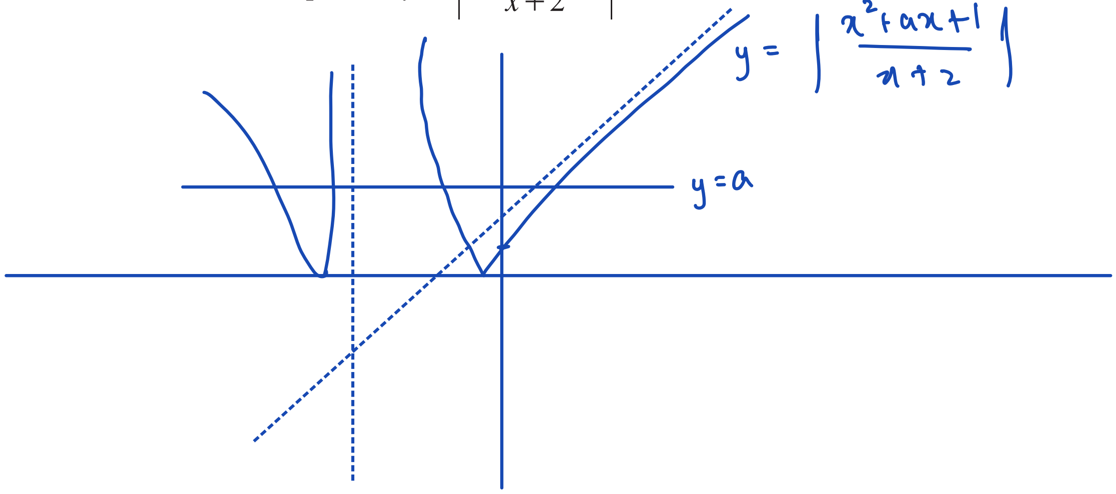
$y = \frac{x^2+ax+1}{x+2}$ $x=0$ $y=\frac{1}{2}$

(c) Sketch C, stating the coordinates of the point of intersection with the y-axis and labelling the asymptotes. [3]



Intercept: $(0, \frac{1}{2})$

(d) (i) Sketch the curve with equation $y = \left|\frac{x^2+ax+1}{x+2}\right|$. [2]



(ii) On your sketch in part **(i)**, draw the line $y = a$. [1]

(iii) It is given that $\left|\frac{x^2+ax+1}{x+2}\right| < a$ for $-5-\sqrt{14} < x < -3$ and $-5+\sqrt{14} < x < 3$.

$-8{\cdot}7$ -3

Find the value of a. [2]

$x^2 + \cancel{ax} + 1 = \cancel{ax} + 2a$

$x^2 + (1-2a) = 0$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$x = \frac{\sqrt{-4(1-2a)}}{2}$

$x = \pm\frac{1}{2}\sqrt{8a-4} \Rightarrow \frac{1}{2}(2)\sqrt{2a-1}$

$x = \pm\sqrt{2a-1}$

$x^2+ax+1 = -ax-2a$

$x^2 + 2ax + (1+2a) = 0$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$x = -a \pm \sqrt{a^2-2a-1}$

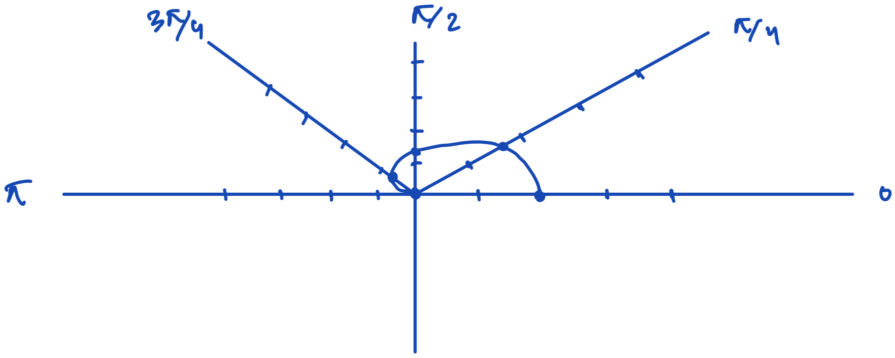
$\Rightarrow -a - \sqrt{a^2-2a-1} < x < -\sqrt{2a-1}$

$\Rightarrow -a + \sqrt{a^2-2a-1} < x < \sqrt{2a-1}$

$\boxed{\therefore a = 5}$

7 The curve C has polar equation $r^2 = (\pi - \theta)\tan^{-1}(\pi - \theta)$, for $0 \leqslant \theta \leqslant \pi$.

(a) Sketch C and state the polar coordinates of the point of C furthest from the pole. [3]



$(\pi \tan^{-1}\pi, 0)$ OR $(1.9916, 0)$

(b) Using the substitution $u = \pi - \theta$, or otherwise, find the area of the region enclosed by C and the initial line. [7]

$A = \frac{1}{2}\int r^2 \, d\theta$ $\quad u = \pi - \theta$

$= \frac{1}{2}\int_0^{\pi} \pi - \theta \tan^{-1}(\pi - \theta) \, d\theta$ $\quad du = -1 \, d\theta$

$= -\frac{1}{2}\int_0^{\pi} -(\pi - \theta)\tan^{-1}(\pi - \theta) \, d\theta$ $\quad u = \tan^{-1}u \quad v' = u$

$= -\frac{1}{2}\int_0^{\pi} u\tan^{-1}u \, du$ $\quad u' = \frac{1}{1+u^2} \quad v = \frac{1}{2}u^2$

$= -\frac{1}{2}\left[\frac{1}{2}u^2\tan^{-1}u - \int_0^{\pi} \frac{u^2}{2(1+u^2)} \, du\right]$

θ	0	π
u	π	0

$= -\frac{1}{4}u^2\tan^{-1}u + \frac{1}{4}\int_0^{\pi} \frac{u^2}{1+u^2} \, du$

$= -\frac{1}{4}u^2\tan^{-1}u + \frac{1}{4}\int_0^{\pi} \frac{(1+u^2)-1}{1+u^2} \, du$

$= -\frac{1}{4}u^2\tan^{-1}u + \frac{1}{4}\int_0^{\pi} 1 - \frac{1}{1+u^2} \, du$

$= \left[-\frac{1}{4}u^2\tan^{-1}u + \frac{1}{4}(u - \tan^{-1}u)\right]_0^{\pi}$

$= \left(-\frac{1}{4}\pi^2\tan^{-1}\pi + \frac{1}{4}(\pi - \tan^{-1}\pi)\right) - \left(\frac{1}{4}(0^2)\tan^{-1}0 + \frac{1}{4}(0 - \tan^{-1}0)\right)$

$= -2.65 \Rightarrow |-2.65|$

$\boxed{A = 2.65}$

$r^2 = (\pi - \theta)\tan^{-1}(\pi - \theta)$

(c) Show that, at the point of C furthest from the initial line,

$$2(\pi-\theta)\tan^{-1}(\pi-\theta)\cot\theta - \frac{\pi-\theta}{1+(\pi-\theta)^2} - \tan^{-1}(\pi-\theta) = 0$$

and verify that this equation has a root for θ between 1.2 and 1.3. [5]

$r^2 = (\pi - \theta)\tan^{-1}(\pi-\theta)$ $\quad$ $y = r\sin\theta \Rightarrow r = \frac{y}{\sin\theta}$

$r = \sqrt{(\pi-\theta)\tan^{-1}(\pi-\theta)}$

$y = \sqrt{(\pi-\theta)\tan^{-1}(\pi-\theta)}\,\sin\theta$

$\left.\frac{dy}{d\theta}\right\}$ find this

$v = \sin\theta$

$v' = \cos\theta$

$u = \left((\pi-\theta)\tan^{-1}(\pi-\theta)\right)^{1/2}$

$u' = \frac{1}{2}\left[(\pi-\theta)\tan^{-1}(\pi-\theta)\right]^{-1/2} \cdot \left[\frac{-(\pi-\theta)}{1+(\pi-\theta)^2} - \tan^{-1}(\pi-\theta)\right]$

$\frac{dy}{d\theta} = \sqrt{(\pi-\theta)\tan^{-1}(\pi-\theta))}\cos\theta - \frac{(\pi-\theta)(1+(\pi-\theta)^2)^{-1} + \tan^{-1}(\pi-\theta)}{2\sqrt{(\pi-\theta)\tan^{-1}(\pi-\theta)}}\sin\theta = 0$

$= \frac{2(\pi-\theta)\tan^{-1}(\pi-\theta)\cot\theta}{2\sqrt{(\pi-\theta)\tan^{-1}(\pi-\theta)}} - \frac{(\pi-\theta)(1+(\pi-\theta)^2) + \tan^{-1}(\pi-\theta)}{2\sqrt{(\pi-\theta)\tan^{-1}(\pi-\theta)}}\frac{\sin\theta}{\sin\theta} = 0$

$= 2(\pi-\theta)\tan^{-1}(\pi-\theta)\cot\theta - \frac{\pi-\theta}{1+(\pi-\theta)^2} - \tan^{-1}(\pi-\theta) = 0$

① $2(\pi-1.2)\tan^{-1}(\pi-1.2)\cot(1.2) - \frac{(\pi-1.2)}{1+(\pi-1.2)^2} - \tan^{-1}(\pi-1.2) = 0.151$

② $2(\pi-1.3)\tan^{-1}(\pi-1.3)\cot(1.3) - \frac{(\pi-1.3)}{1+(\pi-1.3)^2} - \tan^{-1}(\pi-1.3) = -0.295$

The signs change so a root exists b/w 1.2 & 1.3.