Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

* 2 1 5 2 9 1 9 8 9 5 *

FURTHER MATHEMATICS **9231/11**

Paper 1 Further Pure Mathematics 1 **October/November 2024**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages.

DC (CE) 329672/2

1 The matrix **M** represents the sequence of two transformations in the x-y plane given by a stretch parallel to the x-axis, scale factor k ($k \neq 0$), followed by a shear, x-axis fixed, with (0, 1) mapped to (k, 1).

(a) Show that $\mathbf{M} = \begin{pmatrix} k & k \\ 0 & 1 \end{pmatrix}$. [4]

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(b) The transformation represented by **M** has a line of invariant points.

Find, in terms of k, the equation of this line. [3]

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The unit square S in the x-y plane is transformed by $\mathbf{M}$ onto the parallelogram P.

(c) Find, in terms of k, a matrix which transforms P onto S. [1]

(d) Given that the area of P is $3k^2$ units^2, find the possible values of k. [2]

2 Prove by mathematical induction that, for all positive integers n,

$$\frac{d^n}{dx^n}\left(\tan^{-1}x\right) = P_n(x)\left(1+x^2\right)^{-n},$$

where $P_n(x)$ is a polynomial of degree $n-1$. [6]

3 The quartic equation $x^4 + 2x^3 - 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\alpha^4, \beta^4, \gamma^4, \delta^4$ and state the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. [5]

(b) Find the value of $\alpha^5+\beta^5+\gamma^5+\delta^5$. [3]

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(c) Find the value of $\alpha^8+\beta^8+\gamma^8+\delta^8$. [2]

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4 **(a)** Use the method of differences to find $\sum_{r=1}^{n} \frac{5k}{(5r+k)(5r+5+k)}$ in terms of n and k, where k is a positive constant. [4]

It is given that $\sum_{r=1}^{\infty} \frac{5k}{(5r+k)(5r+5+k)} = \frac{1}{3}$.

(b) Find the value of k. [2]

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(c) Hence find $\sum_{r=n}^{n^2} \frac{5k}{(5r+k)(5r+5+k)}$ in terms of n. [2]

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5 **(a)** Show that the curve with Cartesian equation

$$(x^2+y^2)^2 = 6xy$$

has polar equation $r^2 = 3\sin 2\theta$. [2]

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The curve C has polar equation $r^2 = 3\sin 2\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(b) Sketch C and state the maximum distance of a point on C from the pole. [3]

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(c) Find the area of the region enclosed by C. [2]

(d) Find the maximum distance of a point on C from the initial line. [6]

6 The curve C has equation $y = \dfrac{4x^2 + x + 1}{2x^2 - 7x + 3}$.

(a) Find the equations of the asymptotes of C. [2]

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(b) Find the coordinates of any stationary points on C. [4]

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(c) Sketch C, stating the coordinates of any intersections with the axes. [5]

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(d) Sketch the curve with equation $y = \left|\frac{4x^2 + x + 1}{2x^2 - 7x + 3}\right|$ and state the set of values of k for which $\left|\frac{4x^2 + x + 1}{2x^2 - 7x + 3}\right| = k$ has 4 distinct real solutions. [2]

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7 The lines l_1 and l_2 have equations $\mathbf{r} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} + \mathbf{k})$ and $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 9\mathbf{k} + \mu(\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})$ respectively. The plane Π_1 contains l_1 and is parallel to l_2.

(a) Find the equation of Π_1, giving your answer in the form $ax + by + cz = d$. [4]

The plane Π_2 contains l_2 and the point with coordinates $(2, -1, 7)$.

(b) Find the acute angle between Π_1 and Π_2. [4]

The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2.

(c) Find a vector equation for PQ. [7]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

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FURTHER MATHEMATICS **9231/11**

Paper 1 Further Pure Mathematics 1 **October/November 2024**

MARK SCHEME

Maximum Mark: 75

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the October/November 2024 series for most Cambridge IGCSE, Cambridge International A and AS Level components, and some Cambridge O Level components.

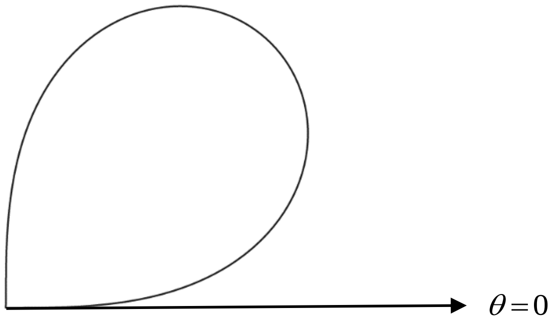
Question	Answer	Marks	Guidance
1(a)	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$	**B1**	[Stretch parallel to the x-axis, scale factor k $(k \neq 0)$]. (Allow without identification.)
	$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$	**B1**	[Shear, x-axis fixed, with $(0,1)$ mapped to $(k,1)$.] (Allow without identification.)
	$\mathbf{M} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} k & k \\ 0 & 1 \end{pmatrix}$	**M1 A1**	Correct order for M1, must have identified which matrix gives which transformation, AG.
		4	
1(b)	$\begin{pmatrix} k & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx+ky \\ y \end{pmatrix}$	**B1**	Transforms $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} X \\ Y \end{pmatrix}$
	$kx + ky = x$	**M1**	Sets $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$
	$y = \frac{1-k}{k}x$ oe	**A1**	
		3	
1(c)	$\mathbf{M}^{-1} = \frac{1}{k}\begin{pmatrix} 1 & -k \\ 0 & k \end{pmatrix}$	**B1**	(An alternative is possible.)
		1	

Question	Answer	Marks	Guidance
1(d)	$\|k\| = 3k^2$	M1	Uses that $\det \mathbf{M} = k$. Without modulus is SC B1.
	$k \neq 0 \Rightarrow k = \pm\frac{1}{3}$	A1	
		2	

Question	Answer	Marks	Guidance
2	$\frac{\mathrm{d}\left(\tan^{-1} x\right)}{\mathrm{d}x} = \frac{1}{1+x^2}$ so true when $n = 1$.	B1	Differentiates once.
	Assume that $\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\tan^{-1} x\right) = \mathrm{P}_k(x)\left(1+x^2\right)^{-k}$, where $\deg \mathrm{P}_k(x) = k-1$.	B1	States inductive hypothesis. Must have $\deg \mathrm{P}_k(x) = k-1$.
	$\frac{\mathrm{d}^{k+1}\left(\tan^{-1} x\right)}{\mathrm{d}x^{k+1}} = \mathrm{P}_k{}'(x)\left(1+x^2\right)^{-k} - 2kx\mathrm{P}_k(x)\left(1+x^2\right)^{-k-1}$	M1 A1	Differentiates kth derivative using the product rule.
	$= \left(\mathrm{P}_k{}'(x)\left(1+x^2\right) - 2kx\mathrm{P}_k(x)\right)\left(1+x^2\right)^{-k-1}$ so $\deg \mathrm{P}_{k+1}(x) = k$	A1	Writes in the form $\mathrm{P}_{k+1}(x)\left(1+x^2\right)^{-k-1}$.
	So true when $n = k+1$. By induction, true for all positive integers n.	A1	Attempts to show degree of $\mathrm{P}_{k+1}(x)$ is at most k (condone not showing coefficient of x^k is non-zero) and states conclusion.
		6	

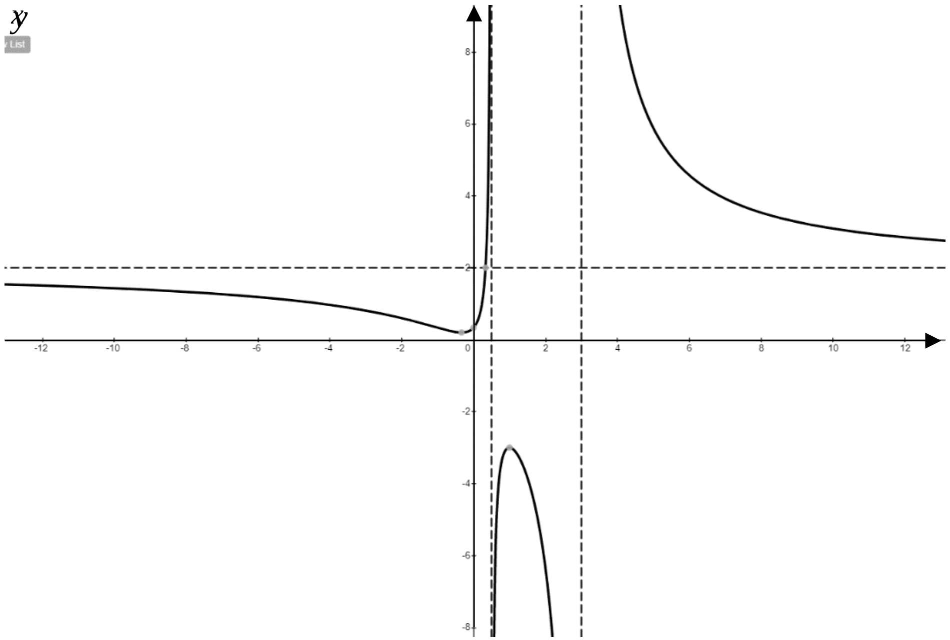
Question	Answer	Marks	Guidance
3(a)	$y = x^4$	B1	Uses correct substitution.
	$y + 2y^{\frac{3}{4}} - 1 = 0 \Rightarrow 16y^3 = (1-y)^4$	M1	Substitutes and obtains an equation not involving radicals.
	$16y^3 = 1 - 4y + 6y^2 - 4y^3 + y^4$	M1	Uses binomial expansion.
	$y^4 - 20y^3 + 6y^2 - 4y + 1 = 0$	A1	Must be an equation.
	$\alpha^4 + \beta^4 + \gamma^4 + \delta^4 = 20$	B1	
		5	
3(b)	$\alpha + \beta + \gamma + \delta = -2$	B1	
	$x^5 + 2x^4 - x = 0 \Rightarrow \alpha^5 + \beta^5 + \gamma^5 + \delta^5 = -2(20) + (-2)$	M1	Multiplies original equation by x and substitutes.
	–42	A1	
		3	
3(c)	$\alpha^8 + \beta^8 + \gamma^8 + \delta^8 = 20^2 - 2(6)$	M1	Uses formula for sum of squares. Or alternative complete method eg another substitution.
	388	A1	CAO.
		2	

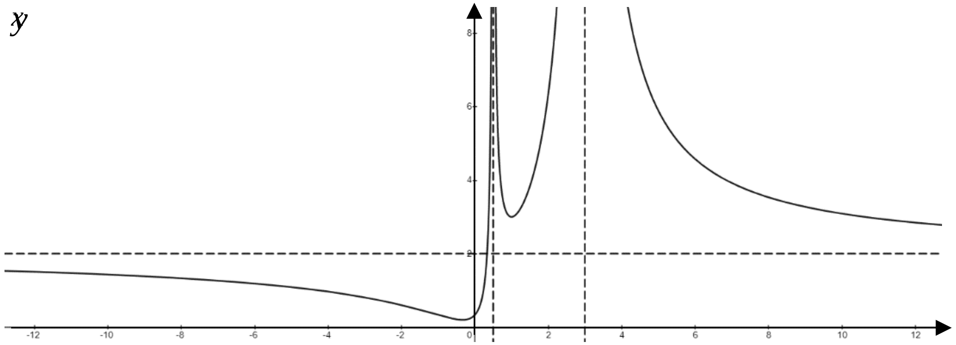
Question	Answer	Marks	Guidance
4(a)	$\frac{5k}{(5r+k)(5r+5+k)}=k\left(\frac{1}{5r+k}-\frac{1}{5r+5+k}\right)$	M1 A1	Finds partial fractions. (Don't allow a substitution of a value for k.)
	$\sum_{r=1}^{n}\frac{5k}{(5r+k)(5r+5+k)}=k\left(\frac{1}{5+k}-\frac{1}{10+k}+\frac{1}{10+k}-\frac{1}{15+k}+\ldots+\frac{1}{5n+k}-\frac{1}{5n+5+k}\right)$	M1	Writes at least three terms, including last. (Allow any value of k.)
	$=k\left(\frac{1}{5+k}-\frac{1}{5n+5+k}\right)$	A1	
		4	
4(b)	$\frac{k}{5+k}=\frac{1}{3}\Rightarrow 3k=5+k\Rightarrow k=\frac{5}{2}$	M1 A1	
		2	
4(c)	$\sum_{r=n}^{n^2}\frac{5k}{(5r+k)(5r+5+k)}=\sum_{r=1}^{n^2}\frac{5k}{(5r+k)(5r+5+k)}-\sum_{r=1}^{n-1}\frac{5k}{(5r+k)(5r+5+k)}$	M1	Or applies the method of differences again.
	$=k\left(\frac{1}{5+k}-\frac{1}{5n^2+5+k}\right)-k\left(\frac{1}{5+k}-\frac{1}{5n+k}\right)=k\left(\frac{1}{5n+k}-\frac{1}{5n^2+5+k}\right)$ $=\frac{1}{2n+1}-\frac{1}{2n^2+3}$	A1FT	FT on their value of k (must be substituted in).
		2	

Question	Answer	Marks	Guidance
5(a)	$r^4 = 6r^2 \sin\theta\cos\theta$	**M1**	Substitutes $x = r\cos\theta$, $y = r\sin\theta$ and applies $\sin 2\theta = 2\sin\theta\cos\theta$.
	$r^2 = 3\sin 2\theta$	**A1**	AG.
		2	
5(b)		**B1**	Correct position and symmetrical about $\theta = \frac{1}{4}\pi$.
		B1	Single correct loop.
	$\sqrt{3}$	**B1**	States maximum distance or labels sketch. Allow $\left(\sqrt{3}, \frac{1}{4}\pi\right)$ but not $\left(\frac{1}{4}\pi, \sqrt{3}\right)$. Allow 3sf.
		3	
5(c)	$\frac{3}{2}\int_0^{\frac{\pi}{2}} \sin 2\theta \, d\theta = \frac{3}{2}\left[-\frac{1}{2}\cos 2\theta\right]_0^{\frac{\pi}{2}}$	**M1**	Forms $\frac{1}{2}\int r^2 \, d\theta$. (Allow with wrong limits.)
	$\frac{3}{2}$	**A1**	
		2	

Question	Answer	Marks	Guidance
5(d)	$y = 3^{\frac{1}{2}} \sin^{\frac{1}{2}} 2\theta \sin\theta$	B1	
	$\sin^{\frac{1}{2}} 2\theta \cos\theta + \sin^{-\frac{1}{2}} 2\theta \cos 2\theta \sin\theta = 0$	M1 A1	Sets $\frac{dy}{d\theta} = 0$.
	$\sin 2\theta \cos\theta + \cos 2\theta \sin\theta = 0 \Rightarrow \tan 2\theta = -\tan\theta \Rightarrow \frac{2\tan\theta}{1-\tan^2\theta} = -\tan\theta$	M1	Applies suitable trigonometric identity. Accept $\sin 3\theta = 0$.
	$\theta = \frac{1}{3}\pi$	A1	
	$\frac{3^{\frac{5}{4}}}{2^{\frac{3}{2}}} = 1.40$	A1	AEF.
		6	

Question	Answer	Marks	Guidance
6(a)	$x = \frac{1}{2},\ x = 3$	B1	Vertical asymptotes.
	$y = 2$	B1	Horizontal asymptote.
		2	

Question	Answer	Marks	Guidance
6(b)	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(2x^2-7x+3)(8x+1)-(4x^2+x+1)(4x-7)}{\left(2x^2-7x+3\right)^2}$	M1	Finds $\frac{\mathrm{d}y}{\mathrm{d}x}$. Allow top line only for M1.
	$-3x^2+2x+1=0$	M1	Sets equal to 0 and forms equation.
	$\left(-\frac{1}{3},\frac{1}{5}\right)$, $(1,-3)$	A1 A1	
		4	
6(c)		B1	Axes and asymptotes. Clear identification (label or clear intersection with axes at correct place).
		B1	$x>3$ correctly approaching asymptotes, not too truncated.
		B1	$\frac{1}{2}<x<3$ correct.
		B1	$x<\frac{1}{2}$ correct.
	$\left(0,\frac{1}{3}\right)$	B1	States coordinates of intersection with axis. May be seen on their graph.
		5	

Question	Answer	Marks	Guidance
6(d)		B1FT	FT from sketch in (c). At least two branches.
	$k > 3$	B1	
		2	

Question	Answer	Marks	Guidance
7(a)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 1 & -4 & 2 \end{vmatrix} = \begin{pmatrix} 6 \\ -3 \\ -9 \end{pmatrix} \sim \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$	M1 A1	Finds common perpendicular. Allow one error.
	$-2(1)+(3)+3(-2)=-5$	M1	Substitutes point on l_1.
	$2x - y - 3z = 5$	A1	CAO.
		4	

Question	Answer	Marks	Guidance
7(b)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 2 \\ 1 & 1 & -2 \end{vmatrix} = \begin{pmatrix} 6 \\ 4 \\ 5 \end{pmatrix}$	**M1 A1**	Finds the normal to Π_2.
	$\begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 4 \\ 5 \end{pmatrix} = \sqrt{14}\sqrt{77}\cos\theta \Rightarrow \cos\theta = \frac{-7}{\sqrt{14}\sqrt{77}}$	**M1**	Uses dot product of normal vectors.
	77.7°	**A1**	No ISW. Accept 1.36 rad.
		4	

Question	Answer	Marks	Guidance
7(c)	$\overrightarrow{OP}=\begin{pmatrix}1+2\lambda\\3+\lambda\\-2+\lambda\end{pmatrix}, \overrightarrow{OQ}=\begin{pmatrix}1+\mu\\-2-4\mu\\9+2\mu\end{pmatrix} \Rightarrow \overrightarrow{PQ}=\begin{pmatrix}\mu-2\lambda\\-5-4\mu-\lambda\\11+2\mu-\lambda\end{pmatrix}$	**M1 A1**	Finds $\overrightarrow{PQ}$.
	$\begin{pmatrix}\mu-2\lambda\\-5-4\mu-\lambda\\11+2\mu-\lambda\end{pmatrix}\bullet\begin{pmatrix}2\\1\\1\end{pmatrix}=0$ or $\begin{pmatrix}\mu-2\lambda\\-5-4\mu-\lambda\\11+2\mu-\lambda\end{pmatrix}=k\begin{pmatrix}6\\-3\\-9\end{pmatrix}$	**M1**	Uses that dot product of $\overrightarrow{PQ}$ with line direction is zero, or, alternatively, $\overrightarrow{PQ}$ is a multiple of the common perpendicular.
	$-6\lambda+6=0$	**A1**	Deduces one equation.
	$\begin{pmatrix}\mu-2\lambda\\-5-4\mu-\lambda\\11+2\mu-\lambda\end{pmatrix}\bullet\begin{pmatrix}1\\-4\\2\end{pmatrix}=0 \Rightarrow 21\mu+42=0$	**A1**	Deduces second equation.
	$\lambda=1 \Rightarrow \overrightarrow{OP}=\begin{pmatrix}3\\4\\-1\end{pmatrix}$ or $\mu=-2 \Rightarrow \overrightarrow{OQ}=\begin{pmatrix}-1\\6\\5\end{pmatrix}$	**M1**	Solves for λ or μ and substitutes into $\overrightarrow{OP}$ or $\overrightarrow{OQ}$
	$\mathbf{r}=\begin{pmatrix}-1\\6\\5\end{pmatrix}+t\begin{pmatrix}6\\-3\\-9\end{pmatrix}$	**A1 FT**	OE. FT using their common perpendicular. Must have "$\mathbf{r}=$".
		7	

Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

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FURTHER MATHEMATICS **9231/12**

Paper 1 Further Pure Mathematics 1 **October/November 2024**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
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- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **20** pages. Any blank pages are indicated.

DC (PQ) 337015/1

1 The sequence $u_1, u_2, u_3, \dots$ is such that $u_1 = 4$ and $u_{n+1} = 3u_n - 2$ for $n \geqslant 1$.

Prove by induction that $u_n = 3^n + 1$ for all positive integers n. [5]

2 The line l_1 has equation $\mathbf{r} = \mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} - 4\mathbf{k})$.

The plane Π contains l_1 and is parallel to the vector $2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$.

(a) Find the equation of Π, giving your answer in the form $ax + by + cz = d$. [4]

The line l_2 is parallel to the vector $5\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$.

(b) Find the acute angle between l_2 and Π. [3]

3 It is given that

$$\alpha + \beta + \gamma + \delta = 2,$$
$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 3,$$
$$\alpha^3 + \beta^3 + \gamma^3 + \delta^3 = 4.$$

(a) Find the value of $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$. [2]

(b) Find the value of $\alpha^2\beta + \alpha^2\gamma + \alpha^2\delta + \beta^2\alpha + \beta^2\gamma + \beta^2\delta + \gamma^2\alpha + \gamma^2\beta + \gamma^2\delta + \delta^2\alpha + \delta^2\beta + \delta^2\gamma$. [3]

(c) It is given that α, β, γ, δ are the roots of the equation

$$6x^4 - 12x^3 + 3x^2 + 2x + 6 = 0.$$

(i) Find the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. [3]

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(ii) Find the value of $\alpha^5 + \beta^5 + \gamma^5 + \delta^5$. [2]

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4 The matrices **A**, **B** and **C** are given by

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 5 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 0 & -2 \\ -1 & 3 \\ 0 & 0 \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} -2 & -1 & 1 \\ 1 & 1 & 3 \end{pmatrix}.$$

(a) Show that $\mathbf{CAB} = \begin{pmatrix} 3 & -7 \\ -9 & 3 \end{pmatrix}$. [3]

(b) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **CAB**. [5]

Let $\mathbf{M} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.

(c) Give full details of the transformation represented by **M**. [2]

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(d) Find the matrix **N** such that **NM** = **CAB**. [3]

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5 It is given that $S_n = \sum_{r=1}^{n} u_r$, where $u_r = x^{f(r)} - x^{f(r+1)}$ and $x > 0$.

(a) Find S_n in terms of n, x and the function f. [2]

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(b) Given that $f(r) = \ln r$, find the set of values of x for which the infinite series

$$u_1 + u_2 + u_3 + \dots$$

is convergent and give the sum to infinity when this exists. [3]

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(c) Given instead that $f(r) = 2\log_x r$ where $x \neq 1$, use standard results from the List of formulae (MF19) to find $\sum_{n=1}^{N} S_n$ in terms of N. Fully factorise your answer. [4]

6 The curve C has equation $y = \dfrac{x^2+3}{x^2+1}$.

(a) Show that C has no vertical asymptotes and state the equation of the horizontal asymptote. [2]

(b) Show that $1 < y \leqslant 3$ for all real values of x. [4]

(c) Find the coordinates of any stationary points on C. [2]

(d) Sketch C, stating the coordinates of any intersections with the axes and labelling the asymptote. [3]

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(e) Sketch the curve with equation $y = \dfrac{x^2+1}{x^2+3}$ and find the set of values of x for which $\dfrac{x^2+1}{x^2+3} < \dfrac{1}{2}$. [4]

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7 The curve C_1 has polar equation $r = a(\cos\theta + \sin\theta)$ for $-\frac{1}{4}\pi \leqslant \theta \leqslant \frac{3}{4}\pi$, where a is a positive constant.

(a) Find a Cartesian equation for C_1 and show that it represents a circle, stating its radius and the Cartesian coordinates of its centre. [4]

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(b) Sketch C_1 and state the greatest distance of a point on C_1 from the pole. [3]

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The curve C_2 with polar equation $r = a\theta$ intersects C_1 at the pole and the point with polar coordinates $(a\phi, \phi)$.

(c) Verify that $1.25 < \phi < 1.26$. [2]

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(d) Show that the area of the smaller region enclosed by C_1 and C_2 is equal to

$$\tfrac{1}{2}a^2\left(\tfrac{3}{4}\pi + \tfrac{1}{3}\phi^3 - \phi + \tfrac{1}{2}\cos 2\phi\right)$$

and deduce, in terms of a and ϕ, the area of the larger region enclosed by C_1 and C_2. [7]

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Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

CAMBRIDGE
International Education

Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/12**

Paper 1 Further Pure Mathematics 1 **October/November 2024**

MARK SCHEME

Maximum Mark: 75

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

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Question	Answer	Marks	Guidance
1	$u_1 = 4 = 3^1 + 1$	B1	Shows base case
	Assume that it is true for $n = k$, so $u_k = 3^k + 1$.	B1	States inductive hypothesis.
	Then $u_{k+1} = 3(3^k + 1) - 2 = 3^{k+1} + 1$	M1 A1	Substitutes into recursion formula.
	[So, it is also true for $n = k + 1$]. Hence, by induction, $u_n = 3^n + 1$ for all positive integers.	A1	States conclusion.
		5	

Question	Answer	Marks	Guidance
2(a)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -4 \\ 2 & 5 & -4 \end{vmatrix} = \begin{pmatrix} 24 \\ -4 \\ 7 \end{pmatrix}$	M1 A1	Finds vector perpendicular to the plane.
	$24(1) - 4(3) + 7(-1) = d \Rightarrow 24x - 4y + 7z = 5$	M1 A1	Uses point in the plane.
		4	
2(b)	$\begin{pmatrix} 24 \\ -4 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -5 \\ -2 \end{pmatrix} = \sqrt{641}\sqrt{54}\cos\alpha \Rightarrow \cos\alpha = \dfrac{126}{\sqrt{641}\sqrt{54}}$	M1A1FT	Uses dot product of $5\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$ and their normal.
	Acute angle between l_2 and Π is $90 - \alpha = 42.6^\circ$	A1	0.744 radians
		3	

Question	Answer	Marks	Guidance
3(a)	$\alpha^2+\beta^2+\gamma^2+\delta^2=(\alpha+\beta+\gamma+\delta)^2-2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$ $3=2^2-2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$	**M1**	Substitutes into formula for sum of squares.
	$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=\frac{1}{2}$	**A1**	
		2	
3(b)	$\alpha^2(\beta+\gamma+\delta)+\beta^2(\alpha+\gamma+\delta)+\gamma^2(\alpha+\beta+\delta)+\delta^2(\alpha+\beta+\gamma)$ $=\alpha^2(2-\alpha)+\beta^2(2-\beta)+\gamma^2(2-\gamma)+\delta^2(2-\delta)$	**M1 A1**	Factorises and substitutes.
	$2(3)-4=2$	**A1**	
		3	
3(c)(i)	$6S_4-12S_3+3S_2+2S_1+24=0\Rightarrow 6S_4-12(4)+3(3)+2(2)+24=0$	**M1**	Sums and substitutes.
		A1	
	$S_4=\frac{11}{6}$	**A1**	
		3	
3(c)(ii)	$6S_5-12S_4+3S_3+2S_2+6S_1=0\Rightarrow 6S_5-12\left(\frac{11}{6}\right)+3(4)+2(3)+12=0$	**M1**	Multiplies equation through by *x*, sums and substitutes. $S_5-2S_4+\frac{1}{2}S_3+\frac{1}{3}S_2+S_1=0$ $\Rightarrow S_5-2\left(\frac{11}{6}\right)+\frac{1}{2}(4)+\frac{1}{3}(3)+2=0$
	$S_5=-\frac{4}{3}$	**A1**	
		2	

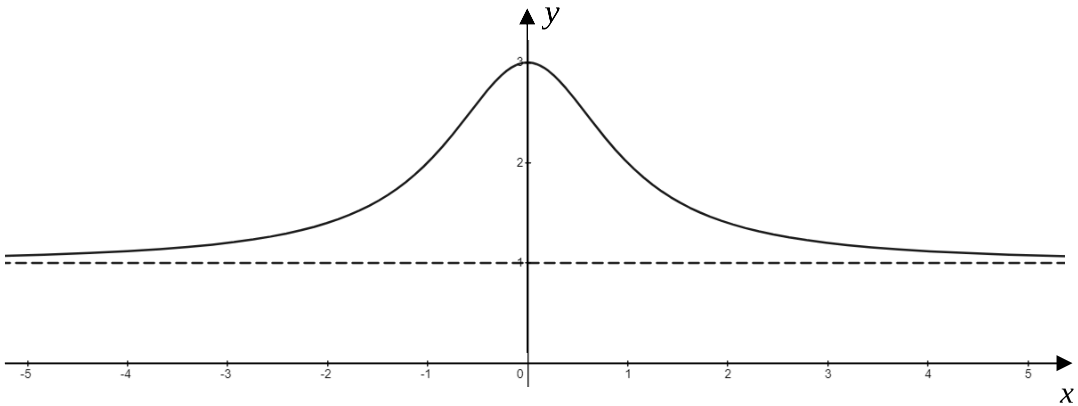
Question	Answer	Marks	Guidance
4(a)	$\begin{pmatrix}-2 & -1 & 1\\ 1 & 1 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\\ 3 & 2 & 5\end{pmatrix}\begin{pmatrix}0 & -2\\ -1 & 3\\ 0 & 0\end{pmatrix}=\begin{pmatrix}-2 & -1 & 1\\ 1 & 1 & 3\end{pmatrix}\begin{pmatrix}-2 & 4\\ -1 & -1\\ -2 & 0\end{pmatrix}$ Or $\begin{pmatrix}-2 & -1 & 1\\ 1 & 1 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\\ 3 & 2 & 5\end{pmatrix}\begin{pmatrix}0 & -2\\ -1 & 3\\ 0 & 0\end{pmatrix}=\begin{pmatrix}-1 & -3 & -4\\ 12 & 9 & 21\end{pmatrix}\begin{pmatrix}0 & -2\\ -1 & 3\\ 0 & 0\end{pmatrix}$	**M1 A1**	Multiplying two matrices correctly, correct dimensions.
	$=\begin{pmatrix}3 & -7\\ -9 & 3\end{pmatrix}$	**B1**	Convincingly completing matrix multiplication, AG.
		3	
4(b)	$\begin{pmatrix}3 & -7\\ -9 & 3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}3x-7y\\ -9x+3y\end{pmatrix}$	**B1**	Transforms $\begin{pmatrix}x\\ y\end{pmatrix}$ to $\begin{pmatrix}X\\ Y\end{pmatrix}$.
	$-9x+3mx=m(3x-7mx)$	**M1 A1**	Uses $y=mx$ and $Y=mX$.
	$-9+3m=3m-7m^2 \Rightarrow 7m^2=9$	**A1**	
	$y=\frac{3}{\sqrt{7}}x$ and $y=-\frac{3}{\sqrt{7}}x$	**A1**	
		5	
4(c)	Stretch	**B1**	
	parallel to the *x*-axis, scale factor 3.	**B1**	
		2	

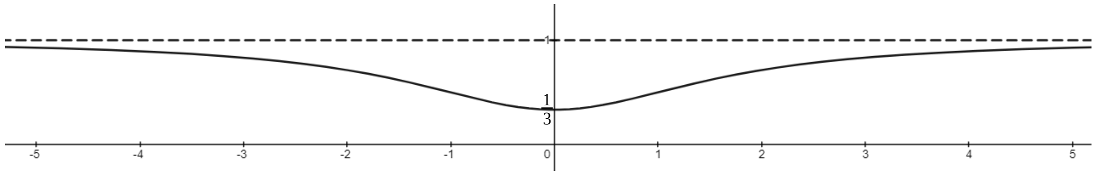
Question	Answer	Marks	Guidance
4(d)	$\mathbf{M}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$	B1	
	$\mathbf{N} = \frac{1}{3}\begin{pmatrix} 3 & -7 \\ -9 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ or $\mathbf{N} = \begin{pmatrix} 3 & -7 \\ -9 & 3 \end{pmatrix}\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}$	M1	Correct order
	$= \begin{pmatrix} 1 & -7 \\ -3 & 3 \end{pmatrix}$	A1	
		3	

Question	Answer	Marks	Guidance
5(a)	$x^{f(1)} - x^{f(2)} + x^{f(2)} - x^{f(3)} + \cdots + x^{f(n)} - x^{f(n+1)}$	M1	Writes at least three terms, including last.
	$= x^{f(1)} - x^{f(n+1)}$	A1	
		2	

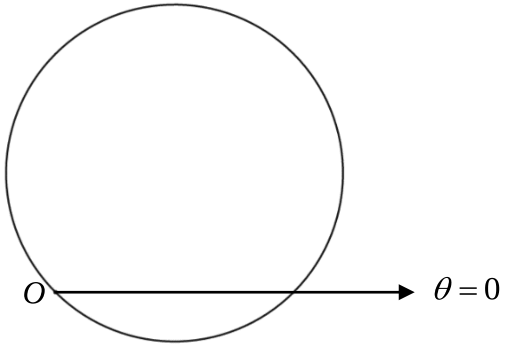
Question	Answer	Marks	Guidance
5(b)	$[0<]x \le 1$	B1	Accept $[0<]x<1$ $\sum_{r=1}^{n} u_r = 1 - x^{\ln(n+1)}$
	$\sum_{r=1}^{\infty} u_r = 1$ [for $x<1$]	B1	Without wrong working.
	$\sum_{r=1}^{\infty} u_r = 0$ for $x=1$	B1	
		3	
5(c)	Uses $x^{2\log_x r} = r^2$	M1	
	$S_n = 1-(n+1)^2 = -n^2 - 2n$	A1	
	$\sum_{n=1}^{N} -n^2 - 2n = -\frac{1}{6}N(N+1)(2N+1) - N(N+1)$	M1	Substitutes formulae from MF19.
	$-\frac{1}{6}N(N+1)(2N+7)$	A1	
		4	

Question	Answer	Marks	Guidance
6(a)	$x^2+1=0$ has no real roots.	**B1**	
	$y=1$	**B1**	Horizontal asymptote.
		2	
6(b)	$yx^2+y=x^2+3 \Rightarrow (1-y)x^2-y+3=0$	**M1 A1**	Forms quadratic in x or uses $y=1+\dfrac{2}{x^2+1}$.
	$-4(1-y)(3-y)\geqslant 0$	**M1**	Uses that discriminant is $\geqslant 0$ or $0<\dfrac{2}{x^2+1}\leqslant 2$.
	$1<y\leq 3$	**A1**	Explanation of why $y\neq 1$ AG.
		4	

Question	Answer	Marks	Guidance
6(c)	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x^2+1)(2x)-(x^2+3)(2x)}{(x^2+1)^2} = 0$	M1	Differentiates
	$(0, 3)$	A1	
	Alternative method for question 6(c)		
	When $y = 3$ $2x^2 = 0$	M1	Using inequality from **(b)**
	$(0, 3)$	A1	
		2	
6(d)		B1	Axes and correct asymptote labelled.
		B1	Correct shape and position.
		B1	States $(0, 3)$ coordinates of intersection with axes, may been seen on diagram.
		3	

Question	Answer	Marks	Guidance
6(e)		B1FT	FT from sketch in (d)
	$\dfrac{x^2+1}{x^2+3}=\dfrac{1}{2}$	M1	Finds critical points.
	$x^2=1 \Rightarrow x=\pm 1$	A1	
	$-1<x<1$	A1	
		4	

Question	Answer	Marks	Guidance
7(a)	$r=a\left(\frac{x}{r}+\frac{y}{r}\right) \Rightarrow r^2=ax+ay$	M1	Uses $x=r\cos\theta$ and $y=r\sin\theta$ to eliminate θ
	$x^2-ax+y^2-ay=0$	M1 A1	OE Obtains Cartesian equation. (using $r^2=x^2+y^2$.)
	$\left(x-\frac{a}{2}\right)^2+\left(y-\frac{a}{2}\right)^2=\frac{a^2}{2} \Rightarrow$ Centre $\left(\frac{a}{2},\frac{a}{2}\right)$ and radius $\frac{a}{\sqrt{2}}$	B1	
		4	

Question	Answer	Marks	Guidance
7(b)	 	B1	Circle
		B1	Correct position, passing through O.
	$a\sqrt{2}$	B1	States maximum distance or labels sketch.
		3	
7(c)	$\cos\phi + \sin\phi - \phi = 0$	M1	
	$\cos 1.25 + \sin 1.25 - 1.25 = 0.01$ $\cos 1.26 + \sin 1.26 - 1.26 = -0.002$	A1	Shows sign change.
		2	

Question	Answer	Marks	Guidance
7(d)	$\frac{a^2}{2}\int_0^{\phi}\theta^2\,d\theta+\frac{a^2}{2}\int_{\phi}^{\frac{3}{4}\pi}(\cos\theta+\sin\theta)^2\,d\theta$	**M1**	Uses area formula on both curves.
		A1	Forms area of smaller region enclosed by C_1 and C_2 with correct limits.
	$\frac{a^2}{2}\int_0^{\phi}\theta^2\,d\theta+\frac{a^2}{2}\int_{\phi}^{\frac{3}{4}\pi}1+2\sin\theta\cos\theta\,d\theta$	**M1**	Applies relevant identities for the circle integral to produce integrable form.
	$\frac{a^2}{2}\left[\frac{1}{3}\theta^3\right]_0^{\phi}+\frac{a^2}{2}\left[\theta+\sin^2\theta\right]_{\phi}^{\frac{3}{4}\pi}$	**A1**	For correct integration of the circle part.
	$\frac{a^2\phi^3}{6}+\frac{a^2}{2}\left(\frac{3}{4}\pi+\frac{1}{2}-\phi-\sin^2\phi\right)=\frac{a^2}{2}\left(\frac{\phi^3}{3}+\frac{3}{4}\pi-\phi+\frac{1}{2}\cos 2\phi\right)$	**A1**	Integrates spiral and substitutes correct limits. AG.
	$\frac{a^2}{2}\pi-\frac{a^2}{2}\left(\frac{\phi^3}{3}+\frac{3}{4}\pi-\phi+\frac{1}{2}\cos 2\phi\right)$	**M1**	Forms area of larger region enclosed by C_1 and C_2.
	$\frac{a^2}{2}\left(-\frac{\phi^3}{3}+\frac{1}{4}\pi+\phi-\frac{1}{2}\cos 2\phi\right)$	**A1**	

Question	Answer	Marks	Guidance
7(d)	**Alternative method for question 7(d)**		
	$\frac{a^2}{2}\int_{-\frac{1}{4}\pi}^{\phi}(\cos\theta+\sin\theta)^2\,\mathrm{d}\theta-\frac{a^2}{2}\int_0^{\phi}\theta^2\mathrm{d}\theta$	**M1**	Uses area formula on both curves.
		A1	Forms area of larger region enclosed by C_1 and C_2 with correct limits.
	$\frac{a^2}{2}\int_{-\frac{1}{4}\pi}^{\phi}1+\sin 2\theta\,\mathrm{d}\theta-\frac{a^2}{2}\int_0^{\phi}\theta^2\mathrm{d}\theta$	**M1**	Applies relevant identities for the circle integral to produce integrable form.
	$\frac{a^2}{2}\left[\theta-\frac{1}{2}\cos 2\theta\right]_{-\frac{\pi}{4}}^{\phi}-\frac{a^2}{2}\left[\frac{1}{3}\theta^3\right]_0^{\phi}$	**A1**	For correct integration of the circle part.
	$\frac{a^2}{2}\left(\phi-\frac{1}{2}\cos 2\phi+\frac{1}{4}\pi\right)-\frac{a^2\phi^3}{6}=\frac{a^2}{2}\left(-\frac{\phi^3}{3}+\frac{1}{4}\pi+\phi-\frac{1}{2}\cos 2\phi\right)$	**A1**	Integrates spiral and substitutes correct limits. Answer not given.
	$\frac{a^2}{2}\pi-\frac{a^2}{2}\left(-\frac{\phi^3}{3}+\frac{1}{4}\pi+\phi-\frac{1}{2}\cos 2\phi\right)$	**M1**	Forms area of smaller region enclosed by C_1 and C_2.
	$\frac{a^2}{2}\left(\frac{\phi^3}{3}+\frac{3}{4}\pi-\phi+\frac{1}{2}\cos 2\phi\right)$	**A1**	AG
		7	

Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

6743059952

FURTHER MATHEMATICS **9231/11**

Paper 1 Further Pure Mathematics 1 **May/June 2024**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages.

DC (SL) 340585

1 The cubic equation $2x^3 + x^2 - px - 5 = 0$, where p is a positive constant, has roots α, β, γ.

(a) State, in terms of p, the value of $\alpha\beta + \beta\gamma + \gamma\alpha$. [1]

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(b) Find the value of $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$. [2]

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(c) Deduce a cubic equation whose roots are $\alpha\beta, \beta\gamma, \alpha\gamma$. [1]

(d) Given that $\alpha^2 + \beta^2 + \gamma^2 = \frac{1}{3}$, find the value of p. [2]

2 Prove by mathematical induction that $6^{4n} + 38^n - 2$ is divisible by 74 for all positive integers n. [6]

3 (a) Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{N} r(r+1)(3r+4) = \tfrac{1}{12}N(N+1)(N+2)(9N+19).$$ [3]

(b) Express $\dfrac{3r+4}{r(r+1)}$ in partial fractions and hence use the method of differences to find

$$\sum_{r=1}^{N} \frac{3r+4}{r(r+1)}\left(\frac{1}{4}\right)^{r+1}$$

in terms of N. [4]

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(c) Deduce the value of $\displaystyle\sum_{r=1}^{\infty} \frac{3r+4}{r(r+1)}\left(\frac{1}{4}\right)^{r+1}$. [1]

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4 The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 14 & 0 \\ 0 & 1 \end{pmatrix}$.

(a) The matrix **M** represents a sequence of two geometrical transformations in the *x*-*y* plane.

Give full details of each transformation, and make clear the order in which they are applied. [4]

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(b) Write $\mathbf{M}^{-1}$ as the product of two matrices, neither of which is **I**. [2]

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(c) Find the equations of the invariant lines, through the origin, of the transformation represented by **M**. [5]

(d) The triangle *ABC* in the *x*-*y* plane is transformed by **M** onto triangle *DEF*.

Given that the area of triangle *DEF* is $28\,\text{cm}^2$, find the area of triangle *ABC*. [2]

5 The points A, B, C have position vectors

$$2\mathbf{i}+2\mathbf{j}+4\mathbf{k}, \qquad 2\mathbf{i}+4\mathbf{j}-\mathbf{k}, \qquad -3\mathbf{i}-3\mathbf{j}+4\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

The point D has position vector $2\mathbf{i}+\mathbf{j}+3\mathbf{k}$.

(b) Find the perpendicular distance from D to the plane ABC. [2]

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(c) Find the shortest distance between the lines AB and CD. [5]

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6 The curve C has equation $y = \frac{x^2 + ax + 1}{x + 2}$, where $a > \frac{5}{2}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Show that C has no stationary points. [4]

(c) Sketch C, stating the coordinates of the point of intersection with the y-axis and labelling the asymptotes. [3]

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(d) **(i)** Sketch the curve with equation $y = \left|\frac{x^2 + ax + 1}{x + 2}\right|$. [2]

(ii) On your sketch in part **(i)**, draw the line $y = a$. [1]

(iii) It is given that $\left|\frac{x^2 + ax + 1}{x + 2}\right| < a$ for $-5 - \sqrt{14} < x < -3$ and $-5 + \sqrt{14} < x < 3$.

Find the value of a. [2]

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7 The curve C has polar equation $r^2 = (\pi - \theta)\tan^{-1}(\pi - \theta)$, for $0 \leqslant \theta \leqslant \pi$.

(a) Sketch C and state the polar coordinates of the point of C furthest from the pole. [3]

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(b) Using the substitution $u = \pi - \theta$, or otherwise, find the area of the region enclosed by C and the initial line. [7]

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(c) Show that, at the point of C furthest from the initial line,

$$2(\pi-\theta)\tan^{-1}(\pi-\theta)\cot\theta-\frac{\pi-\theta}{1+(\pi-\theta)^2}-\tan^{-1}(\pi-\theta)=0$$

and verify that this equation has a root for θ between 1.2 and 1.3. [5]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

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FURTHER MATHEMATICS **9231/11**

Paper 1 Further Pure Mathematics **May/June 2024**

MARK SCHEME

Maximum Mark: 75

Published

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Question	Answer	Marks	Guidance
1(a)	$-\frac{1}{2}p$	**B1**	
		1	
1(b)	$\alpha\beta^2\gamma+\alpha^2\beta\gamma+\alpha\beta\gamma^2=\alpha\beta\gamma(\beta+\gamma+\alpha)=\frac{5}{2}\left(-\frac{1}{2}\right)$	**M1**	Factorises.
	$-\frac{5}{4}$	**A1**	
		2	
1(c)	$4z^3+2pz^2-5z-25=0$	**B1 FT**	OE FT on *their* value of $\alpha\beta+\beta\gamma+\gamma\alpha$ (in terms of p).
		1	
1(d)	$\frac{1}{3}=\frac{1}{4}+p$	**M1**	Uses $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$
	$p=\frac{1}{12}$	**A1**	CAO
		2	

Question	Answer	Marks	Guidance
2	$6^4 + 38 - 2 = 1332$ is divisible by 74.	B1	Checks base case.
	Assume that $6^{4k} + 38^k - 2$ is divisible by 74 for some positive integer k.	B1	States inductive hypothesis.
	Then $6^{4k+4} + 38^{k+1} - 2 = (1295+1) \times 6^{4k} + (37+1) \times 38^k - 2$	M1 A1	Separates $6^{4k} + 38^k - 2$ or considers difference.
	Is divisible by 74 because $1295 \times 6^{4k} + 37 \times 38^k$ is divisible by 74.	A1	
	Hence, by induction, $6^{4k} + 38^k - 2$ is divisible by 74, is true for every positive integer n.	A1	
		6	

Question	Answer	Marks	Guidance
3(a)	$\sum_{r=1}^{N} r(r+1)(3r+4) = 3\sum_{r=1}^{N} r^3 + 7\sum_{r=1}^{N} r^2 + 4\sum_{r=1}^{N} r$	B1	Expands.
	$\frac{3}{4}N^2(N+1)^2 + \frac{7}{6}N(N+1)(2N+1) + 2N(N+1)$	M1	Substitutes formulae from MF19.
	$\frac{1}{12}N\left(9N\left(N^2+2N+1\right)+14\left(2N^2+3N+1\right)+24N+24\right)$ $\frac{1}{12}N\left(9N^3+46N^2+75N+38\right) = \frac{1}{12}N(N+1)(N+2)(9N+19)$	A1	AG
		3	

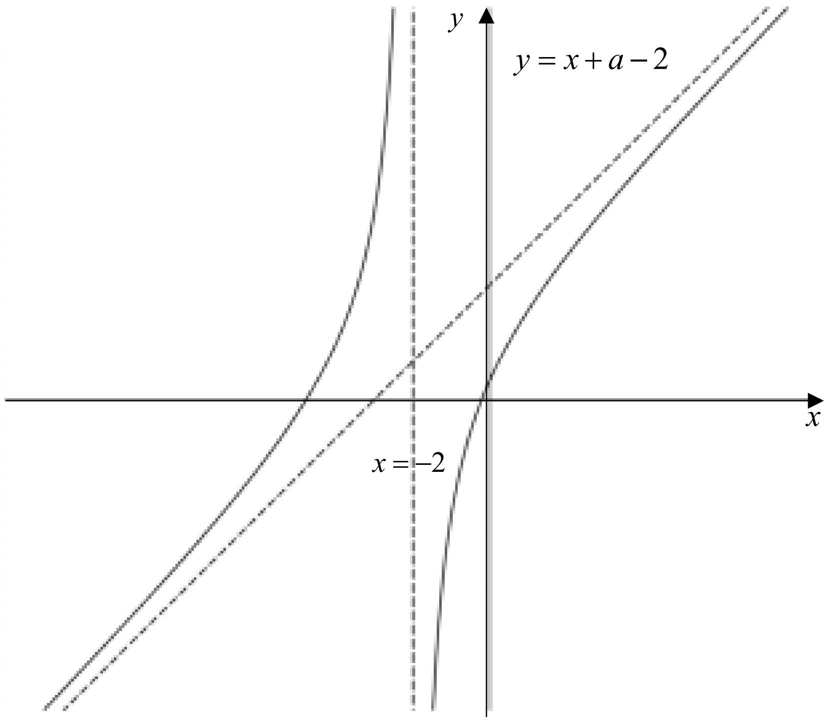
Question	Answer	Marks	Guidance
3(b)	$\frac{3r+4}{r(r+1)}=\frac{4}{r}-\frac{1}{r+1}$	**M1 A1**	Finds partial fractions.
	$\sum_{r=1}^{N}\frac{3r+4}{r(r+1)}\left(\frac{1}{4}\right)^{r+1}=\left(\frac{1}{4^2}\right)\left(\frac{4}{1}-\frac{1}{2}\right)+\left(\frac{1}{4^3}\right)\left(\frac{4}{2}-\frac{1}{3}\right)+\ldots+\left(\frac{1}{4^{N+1}}\right)\left(\frac{4}{N}-\frac{1}{N+1}\right)$	**M1**	Writes at least three terms, including last.
	$\frac{1}{4}-\frac{1}{4^{N+1}(N+1)}$	**A1**	
		4	
3(c)	$\frac{1}{4}$	**B1**	CAO
		1	

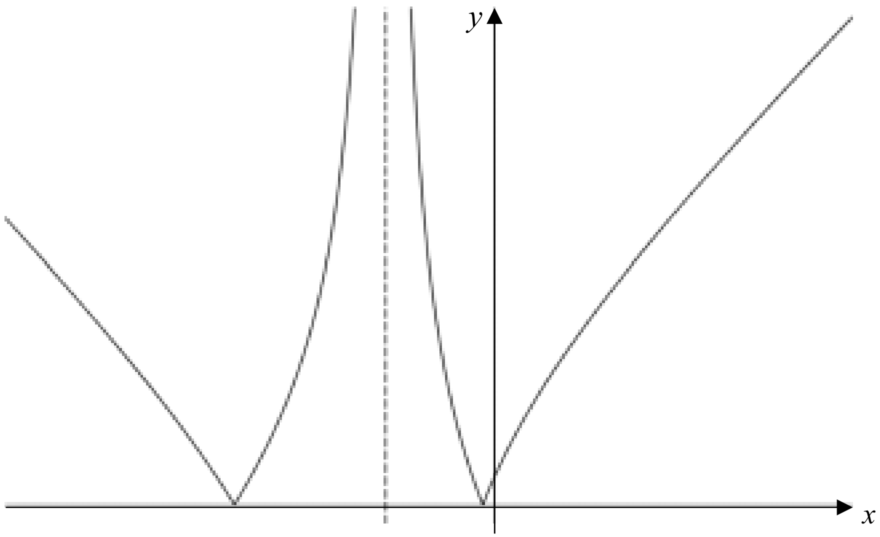
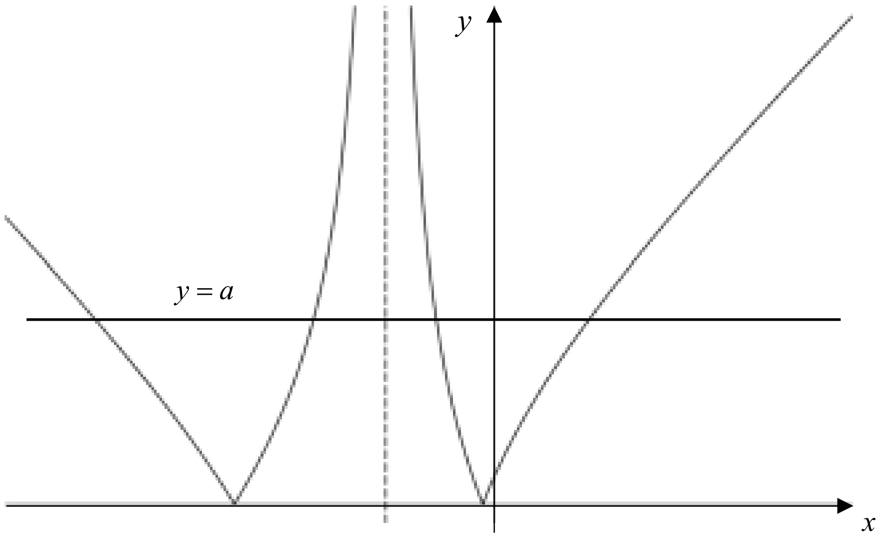
Question	Answer	Marks	Guidance
4(a)	[One way] stretch, rotation	**B1**	
	Stretch followed by rotation	**B1**	Correct order.
	Stretch parallel to the x-axis, scale factor 14	**B1**	
	Rotation, $\frac{1}{3}\pi$ [anticlockwise] about the origin.	**B1**	
		4	

Question	Answer	Marks	Guidance
4(b)	$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}$, $\begin{pmatrix} 14 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \frac{1}{14}\begin{pmatrix} 1 & 0 \\ 0 & 14 \end{pmatrix}$	B1	Both inverses correct.
	$\mathbf{M}^{-1} = \begin{pmatrix} \frac{1}{14} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}$	B1	Correct order.
		2	
4(c)	$\mathbf{M} = \begin{pmatrix} 7 & -\frac{1}{2}\sqrt{3} \\ 7\sqrt{3} & \frac{1}{2} \end{pmatrix}$	B1	
	$\begin{pmatrix} 7 & -\frac{1}{2}\sqrt{3} \\ 7\sqrt{3} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7x - \frac{1}{2}\sqrt{3}y \\ 7\sqrt{3}x + \frac{1}{2}y \end{pmatrix}$	B1	Transforms $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} X \\ Y \end{pmatrix}$
	$7\sqrt{3}x + \frac{1}{2}mx = m\left(7x - \frac{1}{2}\sqrt{3}mx\right)$	M1	Uses $y = mx$ and $Y = mX$.
	$7\sqrt{3} + \frac{1}{2}m = 7m - \frac{1}{2}\sqrt{3}m^2 \Rightarrow \frac{1}{2}\sqrt{3}m^2 - \frac{13}{2}m + 7\sqrt{3} = 0$	A1	
	$y = 2\sqrt{3}x$ and $y = \frac{7}{3}\sqrt{3}x$	A1	
		5	
4(d)	$28 = 14 \times \lvert ABC \rvert$	M1	Uses $\lvert DEF \rvert = \lvert \det \mathbf{M} \rvert \lvert ABC \rvert$.
	2 cm^2	A1	Allow with units missing.
		2	

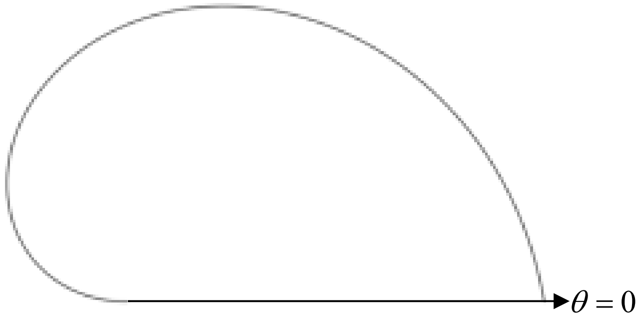
Question	Answer	Marks	Guidance
5(a)	$\overrightarrow{AB} = 2\mathbf{j} - 5\mathbf{k}$ $\quad \overrightarrow{AC} = -5\mathbf{i} - 5\mathbf{j}$ $\quad \overrightarrow{BC} = -5\mathbf{i} - 7\mathbf{j} + 5\mathbf{k}$	B1	Finds direction vectors of two lines in the plane.
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & -5 \\ -5 & -5 & 0 \end{vmatrix} = \begin{pmatrix} -25 \\ 25 \\ 10 \end{pmatrix} \sim \begin{pmatrix} -5 \\ 5 \\ 2 \end{pmatrix}$	M1 A1	Finds normal to the plane *ABC*.
	$-5(2) + 5(2) + 2(4) = 8 \Rightarrow -5x + 5y + 2z = 8$	M1 A1	Substitutes point. AEF for final answer.
		5	
5(b)	$\dfrac{8 - (2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \cdot (-5\mathbf{i} + 5\mathbf{j} + 2\mathbf{k})}{\sqrt{5^2 + 5^2 + 2^2}} = \dfrac{7}{\sqrt{54}} = 0.953$	M1 A1	Correct formula for distance from point to plane.
		2	
5(c)	$\overrightarrow{CD} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -3 \\ -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix}$	B1	
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & -5 \\ 5 & 4 & -1 \end{vmatrix} = \begin{pmatrix} 18 \\ -25 \\ -10 \end{pmatrix}$	M1 A1	Find common perpendicular.
	$\dfrac{1}{\sqrt{1049}} \left\| \begin{pmatrix} -5 \\ -5 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 18 \\ -25 \\ -10 \end{pmatrix} \right\| = \dfrac{35}{\sqrt{1049}} = 1.08$	M1 A1	Uses formula for shortest distance.
		5	

Question	Answer	Marks	Guidance
6(a)	$x=-2$	B1	
	$y=\dfrac{(x+2)(x+a-2)-2a+5}{x+2}=x+a-2+\dfrac{5-2a}{x+2}$	M1	
	$y=x+a-2$	A1	
		3	
6(b)	$\dfrac{dy}{dx}=\dfrac{(x+2)(2x+a)-(x^2+ax+1)}{(x+2)^2}$	M1	Differentiates.
	$x^2+4x+2a-1=0$ $\left(\text{or } \dfrac{dy}{dx}=1+\dfrac{2a-5}{(x+2)^2}\right)$	A1	Forms quadratic equation or simplifies $\dfrac{dy}{dx}$. Not from wrong working.
	$16-4(2a-1)=20-8a<0$ (or $y'>0$) $\Rightarrow$ No stationary points	M1 A1	Consideration of discriminant or sign of y' with correct conclusion.
		4	

Question	Answer	Marks	Guidance
6(c)	 	**B1**	Axes and asymptotes labelled.
		B1	Branches correct. (Asymptotes may cross above, on or below the x-axis.)
	$\left(0,\ \frac{1}{2}\right)$	**B1**	May be seen on their diagram.
		3	

Question	Answer	Marks	Guidance
6(d)(i)		B1	FT from their attempt in part **6(c).**
		B1	Everything correct (approach to vertical asymptote and cusps correct).
		2	
6(d)(ii)		B1	
		1	

Question	Answer	Marks	Guidance
6(d)(iii)	$\dfrac{x^2+ax+1}{x+2}=a$ or $\dfrac{x^2+ax+1}{x+2}=-a$ $x^2+1-2a=0$ or $x^2+2ax+1+2a=0$ $-a-\sqrt{a^2-2a-1}<x<-\sqrt{2a-1},\ -a+\sqrt{a^2-2a-1}<x<\sqrt{2a-1}$	M1	Or direct use of $x=\pm 3$
	$a=5$	A1	
		2	

Question	Answer	Marks	Guidance
7(a)		B1	Correct shape.
		B1	Section $\frac{1}{2}\pi \leqslant \theta \leqslant \pi$ correct.
	$\left(\sqrt{\pi \tan^{-1}\pi},\ 0\right)=(1.99,\ 0)$	B1	May be seen on their diagram.
		3	

Question	Answer	Marks	Guidance
7(b)	$A=\frac{1}{2}\int_0^{\pi}(\pi-\theta)\tan^{-1}(\pi-\theta)\,d\theta$ or $A=-\frac{1}{2}\int_{\pi}^{0}(u)\tan^{-1}(u)\,du$	M1	Uses correct formula. or $A=\frac{1}{2}\int_0^{\pi}(u)\tan^{-1}(u)\,du$
	$\frac{1}{2}\left[\frac{1}{2}(u)^2\tan^{-1}(u)\right]_0^{\pi}-\frac{1}{4}\int_0^{\pi}\frac{u^2}{1+u^2}\,du$	M1 A1	Integrates by parts.
	$\int_0^{\pi}\frac{u^2}{1+u^2}\,du=\int_0^{\pi}\frac{1+u^2-1}{1+u^2}\,du=\int_0^{\pi}1-\frac{1}{1+u^2}\,du$	M1	Rearranges integral into form which can be integrated.
	$\left[u-\tan^{-1}u\right]_0^{\pi}$	A1	
	$A=\frac{1}{2}\left[\frac{1}{2}(u)^2\tan^{-1}(u)\right]_0^{\pi}-\frac{1}{4}\left[u-\tan^{-1}(u)\right]_0^{\pi}$	A1	
	$A=\frac{1}{4}\pi^2\tan^{-1}\pi-\frac{1}{4}\left(\pi-\tan^{-1}\pi\right)=\frac{1}{4}\left(\pi^2+1\right)\tan^{-1}\pi-\frac{1}{4}\pi=2.65$	A1	
		7	

Question	Answer	Marks	Guidance
7(c)	$y=\sqrt{(\pi-\theta)\tan^{-1}(\pi-\theta)}\sin\theta$	B1	Uses $y = r\sin\theta$
	$\frac{dy}{d\theta}=\sqrt{(\pi-\theta)\tan^{-1}(\pi-\theta)}\cos\theta-\frac{(\pi-\theta)\left(1+(\pi-\theta)^2\right)^{-1}+\tan^{-1}(\pi-\theta)}{2\sqrt{(\pi-\theta)\tan^{-1}(\pi-\theta)}}\sin\theta$	M1 A1	Finds derivative.
	$[\theta\neq 0],\pi\Rightarrow 2(\pi-\theta)\tan^{-1}(\pi-\theta)\cot\theta-\frac{\pi-\theta}{1+(\pi-\theta)^2}-\tan^{-1}(\pi-\theta)=0$	A1	Puts $\frac{dy}{d\theta}=0$ and forms equation. AG.
	$2(\pi-1.2)\tan^{-1}(\pi-1.2)\cot 1.2-\frac{\pi-1.2}{1+(\pi-1.2)^2}-\tan^{-1}(\pi-1.2)=0.151$ and $2(\pi-1.3)\tan^{-1}(\pi-1.3)\cot 1.3-\frac{\pi-1.3}{1+(\pi-1.3)^2}-\tan^{-1}(\pi-1.3)=-0.395$	B1	Shows sign change.
		5	

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FURTHER MATHEMATICS **9231/13**

Paper 1 Further Pure Mathematics 1 **May/June 2024**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **20** pages. Any blank pages are indicated.

DC (CE) 329209/1

1 The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} k & 1 & 0 \\ 6 & 5 & 2 \\ -1 & 3 & -k \end{pmatrix},$$

where k is a real constant.

(a) Show that $\mathbf{A}$ is non-singular. [3]

(b) Given that $\mathbf{A}^{-1} = \begin{pmatrix} 3 & 0 & -1 \\ 1 & 0 & 0 \\ -\frac{23}{2} & \frac{1}{2} & 3 \end{pmatrix}$, find the value of k. [2]

2 The cubic equation $x^3 + 2x^2 + 3x + 1 = 0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^2 + 1$, $\beta^2 + 1$, $\gamma^2 + 1$. [3]

(b) Find the value of $(\alpha^2+1)^2+(\beta^2+1)^2+(\gamma^2+1)^2$. [2]

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(c) Find the value of $(\alpha^2+1)^3+(\beta^2+1)^3+(\gamma^2+1)^3$. [2]

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3 The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix}$.

(a) The matrix **M** represents a sequence of two geometrical transformations in the *x*-*y* plane.

Give full details of each transformation, and make clear the order in which they are applied. [4]

(b) Find the equations of the invariant lines, through the origin, of the transformation represented by **M**. [5]

The triangle DEF in the x-y plane is transformed by **M** onto triangle PQR.

(c) Given that the area of triangle PQR is $35\,\text{cm}^2$, find the area of triangle DEF. [2]

4 **(a)** Prove by mathematical induction that, for all positive integers n,

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

[5]

The sum S_n is defined by $S_n = \sum_{r=1}^{n} r^4$.

(b) Using the identity

$$(2r+1)^5 - (2r-1)^5 \equiv 160r^4 + 80r^2 + 2,$$

show that $S_n = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$. [6]

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(c) Find the value of $\lim_{n\to\infty}\left(n^{-5}S_n\right)$. [2]

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5 The lines l_1 and l_2 have equations $\mathbf{r} = \mathbf{i} + 4\mathbf{j} - \mathbf{k} + \lambda(\mathbf{j} - 2\mathbf{k})$ and $\mathbf{r} = -3\mathbf{i} + 4\mathbf{j} + \mu(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ respectively.

(a) Find the shortest distance between l_1 and l_2. [5]

The plane Π_1 contains l_1 and is parallel to l_2.

(b) Obtain an equation of Π_1 in the form $px + qy + rz = s$. [2]

(c) The point $(1, 1, 1)$ lies on the plane Π_2.

It is given that the line of intersection of the planes Π_1 and Π_2 passes through the point $(0, 0, 2)$ and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$.

Obtain an equation of Π_2 in the form $ax + by + cz = d$. [3]

6 The curve C has equation $y = \frac{x+1}{x^2+3}$.

(a) Show that C has no vertical asymptotes and state the equation of the horizontal asymptote. [2]

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(b) Find the coordinates of any stationary points on C. [4]

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(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

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(d) Sketch $y^2 = \frac{x+1}{x^2+3}$, stating the coordinates of the stationary points and the intersections with the axes. [4]

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7 The curve C has polar equation $r^2 = \sin 2\theta \cos \theta$, for $0 \leqslant \theta \leqslant \pi$.

(a) Sketch C and state the equation of the line of symmetry. [3]

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(b) Find a Cartesian equation for C. [3]

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(c) Find the total area enclosed by C. [4]

(d) Find the greatest distance of a point on C from the pole. [6]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

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FURTHER MATHEMATICS **9231/13**

Paper 1 Further Pure Mathematics **May/June 2024**

MARK SCHEME

Maximum Mark: 75

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the May/June 2024 series for most Cambridge IGCSE, Cambridge International A and AS Level and Cambridge Pre-U components, and some Cambridge O Level components.

Question	Answer	Marks	Guidance
1(a)	$k\begin{vmatrix}5 & 2\\3 & -k\end{vmatrix}-\begin{vmatrix}6 & 2\\-1 & -k\end{vmatrix}=k(-5k-6)-(-6k+2)=-5k^2-2$	M1 A1	Evaluates determinant, forms quadratic expression. (Allow 1 slip in the calculation of the determinant for M1 only)
	No real value of $k \Rightarrow$ Non-singular	A1	Convincing conclusion using the discriminant or determinant.
		3	
1(b)	$3k+1=1$ or $-\frac{1}{2}=-\frac{1}{5k^2+2}$	M1	Uses $\mathbf{AA}^{-1}=\mathbf{I}$ or $\det(\mathbf{A}^{-1})=(\det\mathbf{A})^{-1}$ to find equation in k. Could also use a minor determinant e.g. $-k\times 1-3\times 0=0$
	$k=0$	A1	Only $k = 0$, A0 for any additional solutions.
		2	

Question	Answer	Marks	Guidance
2(a)	$y = x^2 + 1$	B1	Uses correct substitution. May use a different letter.
	$(y-1)^{\frac{3}{2}} + 2(y-1) + 3(y-1)^{\frac{1}{2}} + 1 = 0 \Rightarrow (y-1)^{\frac{1}{2}}((y-1)+3) = -2(y-1) - 1$ $(y-1)(y+2)^2 = (-2y+1)^2 \Rightarrow (y-1)(y^2+4y+4) = 4y^2 - 4y + 1$ OR $(x^3+3x)^2 = (-2x^2-1)^2 \Rightarrow x^6 + 2x^4 + 5x^2 - 1 = 0$ $(y-1)^3 + 2(y-1)^2 + 5(y-1) - 1 = 0$	M1	Substitutes and obtains an equation not involving radicals.
	$y^3 - y^2 + 4y - 5 = 0$	A1	OE
		3	
2(b)	$(\alpha^2+1)^2 + (\beta^2+1)^2 + (\gamma^2+1)^2 = 1 - 2(4)$	M1	Uses $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2$ $-2(\alpha\beta+\beta\gamma+\gamma\alpha)$. on *their* answer for part **2(a)**.
	-7	A1	CAO
		2	

Question	Answer	Marks	Guidance
2(c)	$(\alpha^2+1)^3+(\beta^2+1)^3+(\gamma^2+1)^3=-7-4+5(3)$	M1	Uses *their* equation from part **2(a)**.
	4	A1	CAO
	Alternative method for question 2(c)		
	Uses $\sum(\alpha^2+1)^3=$ $\left(\sum(\alpha^2+1)\right)^3-3\left(\sum(\alpha^2+1)\right)\left(\sum(\alpha^2+1)(\beta^2+1)\right)+3\left((\alpha^2+1)(\beta^2+1)(\gamma^2+1)\right)$ $=1^3-3\times1\times4+3\times5$	M1	Uses *their* equation from part **2(a)** in a valid formula.
	4	A1	CAO
		2	

Question	Answer	Marks	Guidance
3(a)	[One-way] stretch, shear	B1	Both types.
	Stretch followed by shear	B1	Correct order.
	Stretch parallel to the x-axis, scale factor 7	B1	
	Shear, x-axis fixed, with $(0,1)$ mapped to $(2,1)$.	B1	
		4	

Question	Answer	Marks	Guidance
3(b)	$\mathbf{M} = \begin{pmatrix} 7 & 2 \\ 0 & 1 \end{pmatrix}$	B1	
	$\begin{pmatrix} 7 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7x+2y \\ y \end{pmatrix}$	B1 FT	Transforms $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} X \\ Y \end{pmatrix}$.
	$mx = m(7x + 2mx)$	M1	Uses $y = mx$ and $Y = mX$.
	$2m^2 + 6m = 0$	A1	
	$y = 0$ and $y = -3x$	A1	SCB1 if M0 and both straight lines correct
		5	
3(c)	Area of PQR = $\lvert 7 \rvert \times$ Area of DEF	M1	
	Area of $DEF = 5\text{ cm}^2$	A1	SCB1 for an answer of 245 (35×7)
		2	

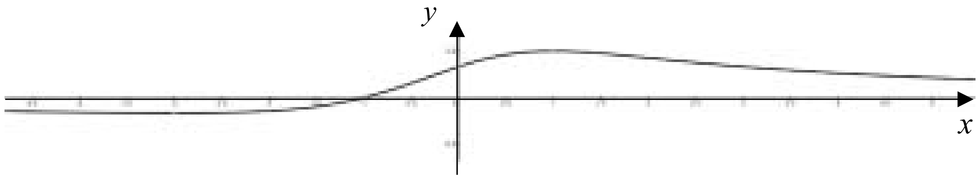
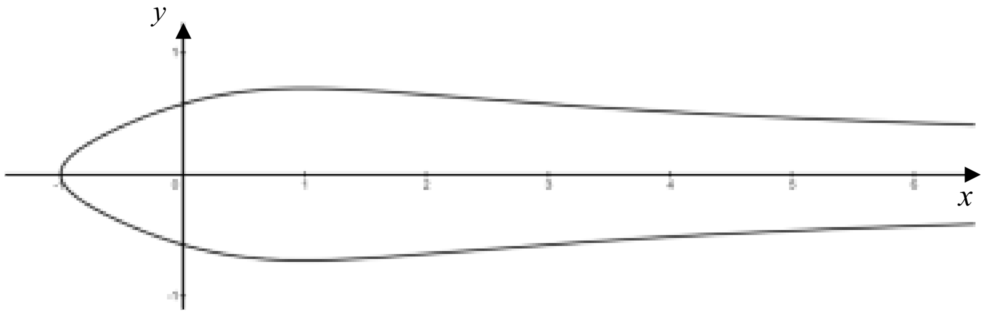
Question	Answer	Marks	Guidance
4(a)	$1^2 = \frac{1}{6}(1)(2)(3)$ so H_1 is true.	B1	Checks base case.
	Assume that $\sum_{r=1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1)$.	B1	States inductive hypothesis.
	$\sum_{r=1}^{k+1} r^2 = \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$	M1	Considers sum to $k+1$.
	$\frac{1}{6}(k+1)\left(2k^2+k+6k+6\right) = \frac{1}{6}(k+1)\left(2k^2+7k+6\right) = \frac{1}{6}(k+1)(k+2)(2k+3)$	A1	
	So H_{k+1} is true. By induction, H_n is true for all positive integers n.	A1	States conclusion.
		5	
4(b)	$3^5 - 1^5 + 5^5 - 3^5 + 7^5 - 5^5 \ldots + (2n+1)^5 - (2n-1)^5$	M1	Applies method of differences to LHS, writes complete terms for at least three values of r including the first and last.
	$(2n+1)^5 - 1$	A1	SCB1 for insufficient complete terms shown.
	$160S_n + \frac{40}{3}n(n+1)(2n+1) + 2n$	M1 A1	Sums RHS and applies standard formulae.
	$160S_n = (2n+1)^5 - \frac{40}{3}n(n+1)(2n+1) - (2n+1)$ $160S_n = (2n+1)\left((2n+1)^4 - \frac{40}{3}n(n+1) - 1\right)$	M1	Makes S_n the subject and factorises.
	$(2n+1)\left(16n^4 + 32n^3 + \frac{32}{3}n^2 - \frac{16}{3}n\right) = \frac{16}{3}n(2n+1)\left(3n^3 + 6n^2 + 2n - 1\right)$ $\Rightarrow S_n = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$	A1	AG
		6	

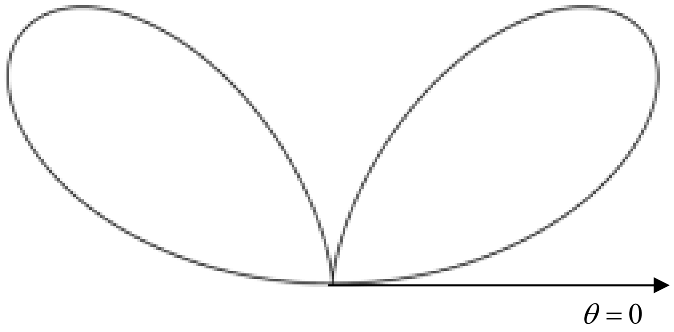
Question	Answer	Marks	Guidance
4(c)	$n^{-5}S_n = \frac{1}{30}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\left(3+\frac{3}{n}-\frac{1}{n^2}\right)$	M1	Divides by n^5.
	$\frac{1}{5}$	A1	
		2	

Question	Answer	Marks	Guidance
5(a)	$\begin{pmatrix}-3\\4\\0\end{pmatrix}-\begin{pmatrix}1\\4\\-1\end{pmatrix}=\begin{pmatrix}-4\\0\\1\end{pmatrix}$	B1	OE. Finds direction of one line to another.
	$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\0 & 1 & -2\\1 & 2 & 1\end{vmatrix}=\begin{pmatrix}5\\-2\\-1\end{pmatrix}$	M1 A1	Find common perpendicular.
	$\frac{1}{\sqrt{30}}\left\lvert\begin{pmatrix}-4\\0\\1\end{pmatrix}\cdot\begin{pmatrix}5\\-2\\-1\end{pmatrix}\right\rvert=\frac{21}{\sqrt{30}}(=3.83)$	M1 A1	Uses formula for shortest distance.
		5	
5(b)	$5(1)-2(4)-1(-1)=-2 \Rightarrow 5x-2y-z=-2$	M1	Uses point in the plane.
		A1 FT	Follow through their normal in part **5(a)**
		2	

Question	Answer	Marks	Guidance
5(c)	$-\mathbf{i}-\mathbf{j}+\mathbf{k}$	**B1**	Finding direction between the two given points.
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & 1 \\ 1 & 4 & -3 \end{vmatrix} = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}$	**M1**	Find common perpendicular. Allow use of their normal from part **(b).**
	$x+2y+3z=6$	**A1**	OE
		3	

Question	Answer	Marks	Guidance
6(a)	$x^2+3=0$ has no real roots.	**B1**	OE
	$y=0$	**B1**	Horizontal asymptote.
		2	
6(b)	$\frac{dy}{dx}=\frac{(x^2+3)-(x+1)(2x)}{(x^2+3)^2}$	**M1**	Finds $\frac{dy}{dx}$.
	$x^2+2x-3=0$	**M1**	Sets equal to 0 and simplifies to quadratic equation.
	$\left(-3,\ -\frac{1}{6}\right),\ \left(1,\ \frac{1}{2}\right)$	**A1 A1**	
		4	

Question	Answer	Marks	Guidance
6(c)		B1	Axes and approach to asymptote.
		B1	Correct smooth shape and position.
	$(-1,\ 0)$, $\left(0,\ \frac{1}{3}\right)$	B1	States coordinates of intersections with axes, may be shown on their graph.
		3	
6(d)		B1	Sections for $x \geqslant 0$ and $x < -1$ correct
		B1	$-1 \leqslant x < 0$ correct.
	$(-1,\ 0)$, $\left(0,\ \pm\frac{1}{\sqrt{3}}\right)$	B1	States coordinates of intersections with axes, may be shown on their graph.
	$\left(1,\ \pm\frac{1}{\sqrt{2}}\right)$	B1	May be shown on their graph.
		4	

Question	Answer	Marks	Guidance
7(a)		B1	One loop correct with initial line.
		B1	Second loop correct with correct form at the pole.
	$\theta = \frac{1}{2}\pi$	B1	May be seen on their diagram.
		3	
7(b)	$r^5 = 2(r\sin\theta)(r\cos\theta)^2$	M1	Use of $\sin 2\theta = 2\sin\theta\cos\theta$, and $x = r\cos\theta$ or $y = r\sin\theta$.
	$r^5 = 2yx^2$	A1	
	$\left(x^2 + y^2\right)^{\frac{5}{2}} = 2x^2 y$	A1	AEF
		3	

Question	Answer	Marks	Guidance
7(c)	$\frac{1}{2}\int_0^{\pi}\sin(2\theta)\cos\theta\,\mathrm{d}\theta$	M1	Applies $\frac{1}{2}\int r^2\,\mathrm{d}\theta$.
	$\frac{1}{2}\int\sin(2\theta)\cos\theta\,\mathrm{d}\theta$	M1	Attempt to integrate in a valid way. May apply $\sin(2\theta)=2\sin\theta\cos\theta$.
	$=\int\sin\theta\cos^2\theta\,\mathrm{d}\theta=-\frac{1}{3}\cos^3\theta\ +\ [c]$	A1	Correct answer (there are alternative forms).
	$=\left[-\frac{1}{3}\cos^3\theta\right]_0^{\pi}=\frac{2}{3}$	A1	CAO
		4	
7(d)	$\frac{\mathrm{d}r}{\mathrm{d}\theta}=\frac{1}{2}(\sin 2\theta\cos\theta)^{-\frac{1}{2}}(2\cos 2\theta\cos\theta-\sin 2\theta\sin\theta)$	*M1 A1	Differentiates with respect to θ.
	$2\cos 2\theta\cos\theta-\sin 2\theta\sin\theta=6\cos^3\theta-4\cos\theta$	dM1	Correct use of relevant identities to express in terms of a single trig function.
	$6\cos^3\theta-4\cos\theta=0\Rightarrow\cos^2\theta=\frac{2}{3}$	dM1 A1	Sets derivative equal to 0 and solves to find one of $\sin^2\theta=\frac{1}{3}$, $\cos^2\theta=\frac{2}{3}$, $\tan^2\theta=\frac{1}{2}$ or θ=0.6155
	$r=\sqrt{\frac{4}{3\sqrt{3}}}=\sqrt{\frac{4\sqrt{3}}{9}}=0.877$	A1	

Question	Answer	Marks	Guidance
7(d)	**Alternative method for question 7(d)**		
	$2r\frac{dr}{d\theta} = 2\left(-2\sin^2\theta\cos\theta + \cos^3\theta\right)$	***M1 A1**	Differentiates [RHS] with respect to θ.
	$2\left(-2\left(1-\cos^2\theta\right)\cos\theta + \cos^3\theta\right) = 6\cos^3\theta - 4\cos\theta$	**dM1**	Correct use of relevant identities in terms of a single trig function
	$6\cos^3\theta - 4\cos\theta = 0 \Rightarrow \cos^2\theta = \frac{2}{3}$	**dM1 A1**	Sets derivative equal to 0 and solves to find one of $\sin^2\theta = \frac{1}{3}$, $\cos^2\theta = \frac{2}{3}$, $\tan^2\theta = \frac{1}{2}$ or θ=0.6155
	$r = \sqrt{\frac{4}{3\sqrt{3}}} = \sqrt{\frac{4\sqrt{3}}{9}} = 0.877$	**A1**	
		6	