Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

4639031271

FURTHER MATHEMATICS **9231/11**

Paper 1 Further Pure Mathematics 1 **October/November 2023**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Any blank pages are indicated.

DC (LK) 311425/1

1 **(a)** By considering $(r+1)^2 - r^2$, use the method of differences to prove that

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1).$$

[4]

(b) Given that $\sum_{r=1}^{n}(r+a) = n$, find a in terms of n. [3]

2 Prove by mathematical induction that, for all positive integers n,

$$1 + 2x + 3x^2 + \ldots + nx^{n-1} = \frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2}.$$ [6]

3 The quartic equation $x^4 + bx^3 + cx^2 + dx - 2 = 0$ has roots α, β, γ, δ. It is given that

$$\alpha + \beta + \gamma + \delta = 3, \qquad \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 5, \qquad \alpha^{-1} + \beta^{-1} + \gamma^{-1} + \delta^{-1} = 6.$$

(a) Find the values of b, c and d. [6]

(b) Given also that $\alpha^3 + \beta^3 + \gamma^3 + \delta^3 = -27$, find the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. [2]

4 The lines l_1 and l_2 have equations

$$\mathbf{r} = -2\mathbf{i} - 3\mathbf{j} - 5\mathbf{k} + \lambda(-4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) \qquad \text{and} \qquad \mathbf{r} = 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \mu(2\mathbf{i} - 3\mathbf{j} + \mathbf{k})$$

respectively.

(a) Find the shortest distance between l_1 and l_2. [5]

The plane Π contains l_1 and the point with position vector $-\mathbf{i}-3\mathbf{j}-4\mathbf{k}$.

(b) Find an equation of Π, giving your answer in the form $ax+by+cz=d$. [4]

5 Let k be a constant. The matrices **A**, **B** and **C** are given by

$$\mathbf{A} = \begin{pmatrix} 1 & k & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 5 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 & -2 \\ -1 & 3 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{C} = \begin{pmatrix} -2 & -1 & 1 \\ 1 & 1 & 3 \end{pmatrix}.$$

It is given that **A** is singular.

(a) Show that $\mathbf{CAB} = \begin{pmatrix} 3 & -7 \\ -9 & 3 \end{pmatrix}$. [5]

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(b) Find the equations of the invariant lines, through the origin, of the transformation in the $x-y$ plane represented by **CAB**. [5]

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(c) The matrices **D**, **E** and **F** represent geometrical transformations in the $x-y$ plane.

- **D** represents an enlargement, centre the origin.
- **E** represents a stretch parallel to the x-axis.
- **F** represents a reflection in the line $y = x$.

Given that $\mathbf{CAB} = \mathbf{D} - 9\mathbf{EF}$, find **D**, **E** and **F**. [5]

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6 **(a)** Show that the curve with Cartesian equation

$$\left(x-\tfrac{1}{2}\right)^2+y^2=\tfrac{1}{4}$$

has polar equation $r=\cos\theta$. [3]

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The curves C_1 and C_2 have polar equations

$$r=\cos\theta \quad \text{and} \quad r=\sin 2\theta$$

respectively, where $0\leqslant\theta\leqslant\frac{1}{2}\pi$. The curves C_1 and C_2 intersect at the pole and at another point P.

(b) Find the polar coordinates of P. [3]

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(c) In a single diagram sketch C_1 and C_2, clearly identifying each curve, and mark the point P. [3]

(d) The region R is enclosed by C_1 and C_2 and includes the line OP.

Find, in exact form, the area of R. [6]

7 The curve C has equation $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \dfrac{x^2+2}{x^2-x-2}$.

(a) Find the equations of the asymptotes of C. [2]

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(b) Find the coordinates of any stationary points on C, giving your answers correct to 1 decimal place. [4]

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(c) Sketch C, stating the coordinates of any intersections with the axes. [3]

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(d) Sketch the curve with equation $y = \frac{1}{f(x)}$. [2]

(e) Find the set of values for which $\frac{1}{f(x)} < f(x)$. [4]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

Cambridge Assessment International Education

Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/11**

Paper 1 Further Pure Mathematics 1 **October/November 2023**

MARK SCHEME

Maximum Mark: 75

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

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Cambridge International is publishing the mark schemes for the October/November 2023 series for most Cambridge IGCSE, Cambridge International A and AS Level components, and some Cambridge O Level components.

This document consists of **15** printed pages.

 [Turn over

Question	Answer	Marks	Guidance
1(a)	$r^2+2r+1-r^2=2r+1$	B1	Expands
	$2\sum_{r=1}^{n} r+n=(n+1)^2-1^2$	M1 A1	Uses method of differences and sums both sides.
	$\Rightarrow 2\sum_{r=1}^{n} r=n^2+n=n(n+1)$	A1	AG.
		4	
1(b)	$\sum_{r=1}^{n}(r+a)=\sum_{r=1}^{n} r+an$	M1	Relates with $\sum r$.
	$\frac{1}{2}n(n+1)+an=n$	M1	Applies $\sum_{r=1}^{n} r=\frac{1}{2}n(n+1)$.
	$a=\frac{1}{2}(1-n)$	A1	
		3	

Question	Answer	Marks	Guidance
2	$1=\frac{1-2x+x^2}{(1-x)^2}=\frac{(1-x)^2}{(1-x)^2}$ so H_1 is true.	**B1**	Checks base case.
	Assume that $\sum_{r=1}^{k} rx^{r-1}=\frac{1-(k+1)x^k+kx^{k+1}}{(1-x)^2}$.	**B1**	States inductive hypothesis.
	$\sum_{r=1}^{k+1} rx^{r-1}=\frac{1-(k+1)x^k+kx^{k+1}}{(1-x)^2}+(k+1)x^k$	**M1**	Considers sum to $k+1$.
	$\frac{1-(k+1)x^k+kx^{k+1}+(k+1)x^k\left(1-2x+x^2\right)}{(1-x)^2}$	**M1**	Puts over a common denominator.
	$\frac{1+kx^{k+1}+(k+1)x^k\left(-2x+x^2\right)}{(1-x)^2}=\frac{1-(k+2)x^{k+1}+(k+1)x^{k+2}}{(1-x)^2}$	**A1**	
	So H_{k+1} is true. By induction, H_n is true for all positive integers n.	**A1**	States conclusion.
		6	

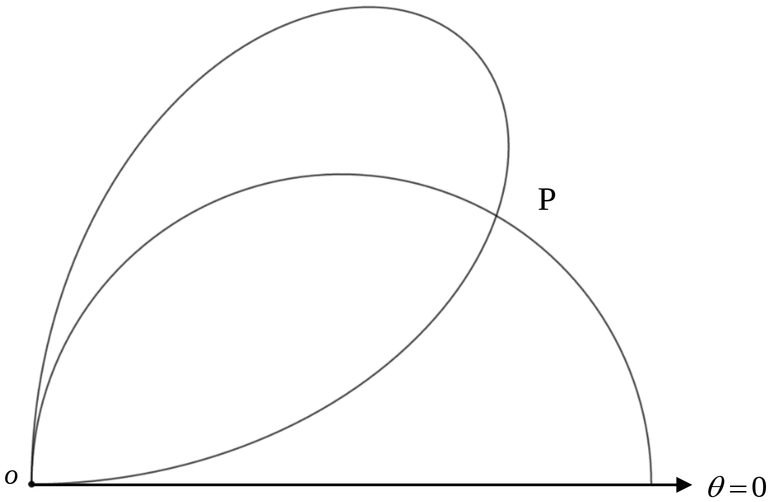
Question	Answer	Marks	Guidance
3(a)	$b = -(\alpha + \beta + \gamma + \delta) = -3$	**B1**	
	$5 = (-3)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$	**M1 A1**	Uses formula for sum of squares.
	$c = 2$	**A1**	
	$6 = \dfrac{\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta}{\alpha\beta\gamma\delta} = \dfrac{-d}{-2}$	**M1**	Uses $\alpha^{-1} + \beta^{-1} + \gamma^{-1} + \delta^{-1} = \dfrac{\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta}{\alpha\beta\gamma\delta}$.
	$d = 12$	**A1**	Equation is $x^4 - 3x^3 + 2x^2 + 12x - 2 = 0$.
		6	
3(b)	$\alpha^4 + \beta^4 + \gamma^4 + \delta^4 = 3(-27) - 2(5) - 12(3) + 2(4)$	**M1**	Uses *their* quartic equation derived in **(a)**.
	–119	**A1**	
		2	

Question	Answer	Marks	Guidance
4(a)	$\begin{pmatrix}2\\-2\\3\end{pmatrix}-\begin{pmatrix}-2\\-3\\-5\end{pmatrix}=\begin{pmatrix}4\\1\\8\end{pmatrix}$	**B1**	Finds direction of one line to another.
	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-4&3&5\\2&-3&1\end{vmatrix}=\begin{pmatrix}18\\14\\6\end{pmatrix}\sim\begin{pmatrix}9\\7\\3\end{pmatrix}$	**M1 A1**	Find common perpendicular.
	$\frac{1}{\sqrt{139}}\left\|\begin{pmatrix}4\\1\\8\end{pmatrix}\cdot\begin{pmatrix}9\\7\\3\end{pmatrix}\right\|=\frac{67}{\sqrt{139}}(=5.68)$	**M1 A1**	Uses formula for shortest distance.
		5	
4(b)	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&1\\-4&3&5\end{vmatrix}=\begin{pmatrix}3\\9\\-3\end{pmatrix}\sim\begin{pmatrix}1\\3\\-1\end{pmatrix}$	**M1 A1**	Finds vector perpendicular to the plane.
	$1(-1)+3(-3)-1(-4)=-6\Rightarrow x+3y-z=-6$	**M1 A1**	Uses point in the plane.
		4	

Question	Answer	Marks	Guidance
5(a)	$\begin{vmatrix}1 & 3\\2 & 5\end{vmatrix} - k\begin{vmatrix}2 & 3\\3 & 5\end{vmatrix} + 3\begin{vmatrix}2 & 1\\3 & 2\end{vmatrix} = 0 \Rightarrow -1-k+3=0 \Rightarrow k=2$	M1 A1	Sets determinant of **A** equal to zero.
	$\begin{pmatrix}-2 & -1 & 1\\1 & 1 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\2 & 1 & 3\\3 & 2 & 5\end{pmatrix}\begin{pmatrix}0 & -2\\-1 & 3\\0 & 0\end{pmatrix} = \begin{pmatrix}-2 & -1 & 1\\1 & 1 & 3\end{pmatrix}\begin{pmatrix}-2 & 4\\-1 & -1\\-2 & 0\end{pmatrix}$	M1	Multiplying two matrices correctly, correct dimensions.
	$\begin{pmatrix}3 & -7\\-9 & 3\end{pmatrix}$	M1 A1	Completing matrix multiplication, AG.
		5	
5(b)	$\begin{pmatrix}3 & -7\\-9 & 3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}3x-7y\\-9x+3y\end{pmatrix}$	B1	Transforms $\begin{pmatrix}x\\y\end{pmatrix}$ to $\begin{pmatrix}X\\Y\end{pmatrix}$.
	$-9x+3mx = m(3x-7mx)$	M1 A1	Uses $y = mx$ and $Y = mX$.
	$-9+3m = 3m-7m^2 \Rightarrow 7m^2 = 9$	A1	
	$y = \frac{3}{\sqrt{7}}x$ and $y = -\frac{3}{\sqrt{7}}x$	A1	
		5	

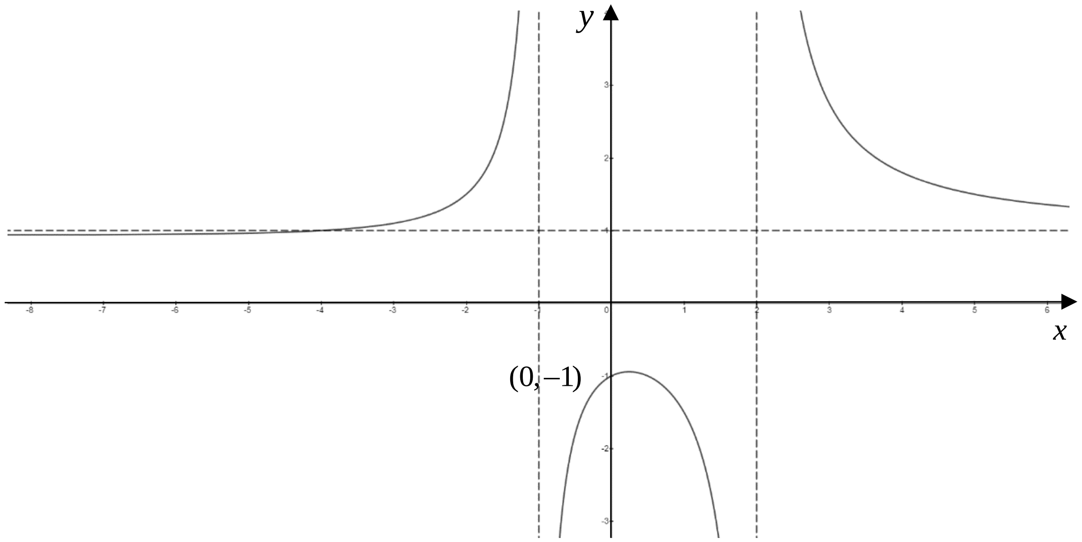
Question	Answer	Marks	Guidance
5(c)	$\mathbf{D} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$	**B1**	
	$\mathbf{E} = \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}$	**B1**	
	$\mathbf{F} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	**B1**	
	$\begin{pmatrix} 3 & -7 \\ -9 & 3 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} - 9\begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix}$	**M1**	Setting up simultaneous equations using their **D** and **E**.
	$\mathbf{D} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ $\mathbf{E} = \begin{pmatrix} \frac{7}{9} & 0 \\ 0 & 1 \end{pmatrix}$	**A1**	Condone $\alpha = 3, \beta = \frac{7}{9}$ if it is clear that they refer to the correct matrices.
		5	

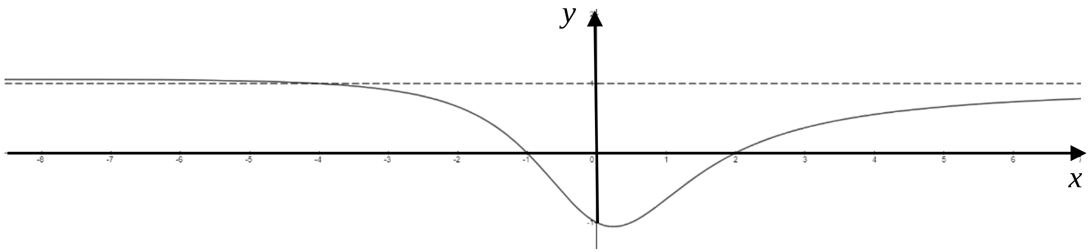
Question	Answer	Marks	Guidance
6(a)	$x^2 - x + \frac{1}{4} + y^2 = \frac{1}{4} \Rightarrow r^2 - r\cos\theta + \frac{1}{4} = \frac{1}{4}$	**B1**	Uses $x^2 + y^2 = r^2$ and $x = r\cos\theta$.
	$r(r - \cos\theta) = 0$	**M1**	Factorises.
	$[r \neq 0 \Rightarrow] r = \cos\theta$	**A1**	AG.
		3	

Question	Answer	Marks	Guidance
6(b)	$\sin 2\theta = \cos\theta \Rightarrow 2\sin\theta\cos\theta = \cos\theta$	M1	Sets r values equal and uses $\sin 2\theta = 2\sin\theta\cos\theta$.
	$\cos\theta \neq 0 \Rightarrow \sin\theta = \frac{1}{2}$	A1	$\cos\theta \neq 0$ must be recognised.
	$\left(\frac{1}{2}\sqrt{3}, \frac{1}{6}\pi\right)$	A1	
		3	
6(c)		B1	Initial line drawn and one curve correct.
		B1	Other curve correct.
		B1	Intersection marked in correct position and both curves labelled.
		3	

Question	Answer	Marks	Guidance
6(d)	$\frac{1}{2}\int_0^{\frac{1}{6}\pi}\sin^2 2\theta\,\mathrm{d}\theta+\frac{1}{2}\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi}\cos^2\theta\,\mathrm{d}\theta$	M1	Uses $\frac{1}{2}\int r^2\,\mathrm{d}\theta$ with correct limits.
	$\frac{1}{2}\int_0^{\frac{1}{6}\pi}\sin^2 2\theta\,\mathrm{d}\theta=\frac{1}{4}\int_0^{\frac{1}{6}\pi}1-\cos 4\theta\,\mathrm{d}\theta$	M1	Integrates $\sin^2 2\theta$ using identity.
	$=\frac{1}{4}\left[\theta-\frac{1}{4}\sin 4\theta\right]_0^{\frac{1}{6}\pi}$	A1	
	$\frac{1}{2}\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi}\cos^2\theta\,\mathrm{d}\theta=\frac{1}{4}\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi}1+\cos 2\theta\,\mathrm{d}\theta$	M1	Integrates $\cos^2\theta$ using identity.
	$=\frac{1}{4}\left[\theta+\frac{1}{2}\sin 2\theta\right]_{\frac{1}{6}\pi}^{\frac{1}{2}\pi}$	A1	
	$\frac{1}{4}\left(\frac{1}{6}\pi-\frac{1}{8}\sqrt{3}\right)+\frac{1}{4}\left(\frac{1}{2}\pi-\frac{1}{6}\pi-\frac{1}{4}\sqrt{3}\right)=\frac{1}{8}\left(\pi-\frac{3}{4}\sqrt{3}\right)$	A1	
		6	

Question	Answer	Marks	Guidance
7(a)	$x=-1,\quad x=2$	B1	Vertical asymptotes.
	$y=1$	B1	Horizontal asymptote.
		2	

Question	Answer	Marks	Guidance
7(b)	$\frac{dy}{dx} = \frac{(x^2 - x - 2)(2x) - (x^2 + 2)(2x - 1)}{(x^2 - x - 2)^2}$	M1*	Finds $\frac{dy}{dx}$.
	$x^2 + 8x - 2 = 0$	DM1	Sets equal to 0 and forms equation.
	$(-8.2, 0.9)$, $(0.2, -0.9)$.	A1 A1	Condone $\left(-4 - 3\sqrt{2}, \frac{2}{3}\sqrt{2}\right)$, $\left(-4 + 3\sqrt{2}, -\frac{2}{3}\sqrt{2}\right)$.
		4	
7(c)		B1	Axes and all three asymptotes.
		B1	Correct shape and position, crossing horizontal asymptote.
		B1	States $(0, -1)$ coordinates of intersection with axes, may be seen on diagram.
		3	

Question	Answer	Marks	Guidance
7(d)		B1 FT	FT from sketch in **(c)**
		B1	All correct.
		2	
7(e)	$\frac{x^2+2}{x^2-x-2}=1$ or $\frac{x^2+2}{x^2-x-2}=-1$ $x+4=0$ or $2x^2-x=0$	M2	Finds critical points, award M1 for each case.
	$x=-4$ or $x=0$, $x=\frac{1}{2}$	A1	
	$-4<x<-1$, $0<x<\frac{1}{2}$, $x>2$	B1	Must have three distinct regions. Condone $\leqslant -1$ and $\geqslant 2$.
		4	

Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

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6289709443

FURTHER MATHEMATICS **9231/12**

Paper 1 Further Pure Mathematics 1 **October/November 2023**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS
- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
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INFORMATION
- The total mark for this paper is 75.
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DC (PQ) 311424/2

1 **(a)** Use standard results from the list of formulae (MF19) to find $\sum_{r=1}^{n}(3r^2+3r+1)$ in terms of n, simplifying your answer. [3]

(b) Show that

$$\frac{1}{r^3} - \frac{1}{(r+1)^3} = \frac{3r^2+3r+1}{r^3(r+1)^3}$$

and hence use the method of differences to find $\sum_{r=1}^{n} \frac{3r^2+3r+1}{r^3(r+1)^3}$. [5]

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(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{3r^2+3r+1}{r^3(r+1)^3}$. [1]

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2 Prove by mathematical induction that, for all positive integers n,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2\mathrm{e}^x\right) = \left(x^2 + 2nx + n(n-1)\right)\mathrm{e}^x .$$ [6]

3 The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, where k is a constant and $k \neq 0$ and $k \neq 1$.

(a) The matrix **M** represents a sequence of two geometrical transformations. State the type of each transformation, and make clear the order in which they are applied. [2]

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The unit square in the $x-y$ plane is transformed by **M** onto parallelogram $OPQR$.

(b) Find, in terms of k, the area of parallelogram $OPQR$ and the matrix which transforms $OPQR$ onto the unit square. [3]

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(c) Show that the line through the origin with gradient $\frac{1}{k-1}$ is invariant under the transformation in the $x-y$ plane represented by **M**. [3]

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4 The cubic equation $27x^3 + 18x^2 + 6x - 1 = 0$ has roots α, β, γ.

(a) Show that a cubic equation with roots $3\alpha + 1$, $3\beta + 1$, $3\gamma + 1$ is

$$y^3 - y^2 + y - 2 = 0.$$ [3]

The sum $(3\alpha+1)^n+(3\beta+1)^n+(3\gamma+1)^n$ is denoted by S_n.

(b) Find the values of S_2 and S_3. [4]

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(c) Find the values of S_{-1} and S_{-2} . [3]

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5 The plane Π_1 has equation $\mathbf{r} = \mathbf{i} - \mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) + \mu(3\mathbf{i} - \mathbf{k})$.

(a) Find an equation for Π_1 in the form $ax + by + cz = d$. [4]

The line l, which does not lie in Π_1, has equation $\mathbf{r} = -3\mathbf{i} + \mathbf{k} + t(\mathbf{i} + \mathbf{j} + \mathbf{k})$.

(b) Show that l is parallel to Π_1. [2]

(c) Find the distance between l and Π_1. [3]

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(d) The plane Π_2 has equation $3x + 3y + 2z = 1$.

Find a vector equation of the line of intersection of Π_1 and Π_2. [4]

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6 The curve C has polar equation $r = e^{-\theta} - e^{-\frac{1}{2}\pi}$, where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) Sketch C and state, in exact form, the greatest distance of a point on C from the pole. [3]

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(b) Find the exact value of the area of the region bounded by C and the initial line. [5]

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(c) Show that, at the point on C furthest from the initial line,

$$1 - e^{\theta - \frac{1}{2}\pi} - \tan\theta = 0$$

and verify that this equation has a root between 0.56 and 0.57. [5]

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7 The curve C has equation $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \dfrac{x^2}{x+1}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Find the coordinates of any stationary points on C. [2]

(c) Sketch C. [3]

(d) Find the coordinates of any stationary points on the curve with equation $y = \dfrac{1}{\mathrm{f}(x)}$. [2]

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(e) Sketch the curve with equation $y = \frac{1}{f(x)}$ and find, in exact form, the set of values for which $\frac{1}{f(x)} > f(x)$. [6]

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FURTHER MATHEMATICS **9231/12**

Paper 1 Further Pure Mathematics 1 **October/November 2023**

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This document consists of **15** printed pages.

Question	Answer	Marks	Guidance
1(a)	$\frac{1}{2}n(n+1)(2n+1)+\frac{3}{2}n(n+1)+n$	M1 A1	Substitutes correct formulae from MF19.
	n^3+3n^2+3n	A1	Simplifies
		3	
1(b)	$\frac{1}{r^3}-\frac{1}{(r+1)^3}=\frac{(r+1)^3-r^3}{r^3(r+1)^3}=\frac{r^3+3r^2+3r+1-r^3}{r^3(r+1)^3}=\frac{3r^2+3r+1}{r^3(r+1)^3}$	M1 A1	Puts over a common denominator and expands, AG.
	$\sum_{r=1}^{n}\frac{3r^2+3r+1}{r^3(r+1)^3}=\sum_{r=1}^{n}\left(\frac{1}{r^3}-\frac{1}{(r+1)^3}\right)$ $=1-\frac{1}{2^3}+\frac{1}{2^3}-\frac{1}{3^3}+\ldots+\frac{1}{n^3}-\frac{1}{(n+1)^3}$	M1 A1	Shows three complete terms, including last.
	$1-\frac{1}{(n+1)^3}$	A1	
		5	
1(c)	1	B1FT	FT from *their* answer to part **(b)**.
		1	

Question	Answer	Marks	Guidance
2	$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\mathrm{e}^x\right)=x^2\mathrm{e}^x+2x\mathrm{e}^x=\left(x^2+2x\right)\mathrm{e}^x$ so true when $n=1$.	**M1 A1**	Differentiates once using the product rule.
	Assume that $\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(x^2\mathrm{e}^x\right)=\left(x^2+2kx+k(k-1)\right)\mathrm{e}^x$ [for some value of k].	**B1**	States inductive hypothesis.
	$\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}\left(x^2\mathrm{e}^x\right)=\left(x^2+2kx+k(k-1)\right)\mathrm{e}^x+\mathrm{e}^x\left(2x+2k\right)$	**M1**	Differentiates *k*th derivative.
	$\left(x^2+2(k+1)x+k(k+1)\right)\mathrm{e}^x$	**A1**	
	So true when $n=k+1$. By induction, true for all positive integers *n*.	**A1**	States conclusion.
		6	

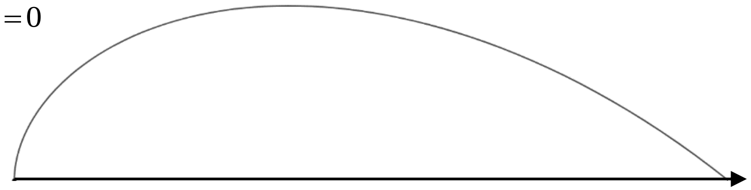
Question	Answer	Marks	Guidance
3(a)	Shear followed by a stretch.	**B2**	Award B1 if given in the wrong order.
		2	
3(b)	$\lvert OPQR\rvert=\lvert\det\mathbf{M}\rvert=\lvert k\rvert$	**B1**	
	$\mathbf{M}^{-1}=\frac{1}{k}\begin{pmatrix}1&0\\-1&k\end{pmatrix}$	**M1 A1**	
		3	

Question	Answer	Marks	Guidance
3(c)	$\begin{pmatrix} k & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ \frac{1}{k-1}x \end{pmatrix}$	B1	Sets $y = \frac{1}{k-1}x$.
	$\begin{pmatrix} k & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ \frac{1}{k-1}x \end{pmatrix} = \begin{pmatrix} kx \\ x + \frac{1}{k-1}x \end{pmatrix} = \begin{pmatrix} kx \\ \frac{k}{k-1}x \end{pmatrix}$	M1	Shows that $Y = \frac{1}{k-1}X$.
	$k\begin{pmatrix} x \\ \frac{1}{k-1}x \end{pmatrix}$	A1	
	Alternative method for 3(c)		
	$\begin{pmatrix} k & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx \\ x+y \end{pmatrix}$	B1	Transforms $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} X \\ Y \end{pmatrix}$
	$X = kx$ and $mX = x + y$ $mkx = x + mx$	M1	Uses $y = mx$ and $Y = mX$
	$m = \dfrac{1}{k-1}$ $y = \dfrac{1}{k-1}x$	A1	AG
		3	

Question	Answer	Marks	Guidance
4(a)	$y = 3x + 1 \Rightarrow x = \frac{1}{3}(y-1)$ $\Rightarrow 27\left(\frac{y-1}{3}\right)^3 + 18\left(\frac{y-1}{3}\right)^2 + 6\left(\frac{y-1}{3}\right) - 1 = 0$	**B1**	Substitutes.
	$\Rightarrow (y-1)^3 + 2(y-1)^2 + 2(y-1) - 1 = 0$ $\Rightarrow y^3 - 3y^2 + 3y - 1 + 2y^2 - 4y + 2 + 2y - 2 - 1 = 0$	**M1**	Expands.
	$y^3 - y^2 + y - 2 = 0$	**A1**	AG.
		3	
4(b)	$S_2 = 1^2 - 2(1) = -1$	**M1 A1**	Uses formula for sum of squares, AG.
	$S_3 = (3\alpha+1)^3 + (3\beta+1)^3 + (3\gamma+1)^3 = -1 - (1) + 6$	**M1**	Uses $y^3 = y^2 - y + 2$ or expands and uses original equation.
	4	**A1**	
		4	
4(c)	$S_{-1} = \frac{(3\alpha+1)(3\beta+1) + (3\beta+1)(3\gamma+1) + (3\gamma+1)(3\alpha+1)}{(3\alpha+1)(3\beta+1)(3\gamma+1)} = \frac{1}{2}$	**B1**	
	$2S_{-2} = S_1 - 3 + S_{-1} = 1 - 3 + \frac{1}{2}$	**M1**	Uses $2y^{-2} = y - 1 + y^{-1}$.
	$S_{-2} = -\frac{3}{4}$	**A1**	CAO
		3	

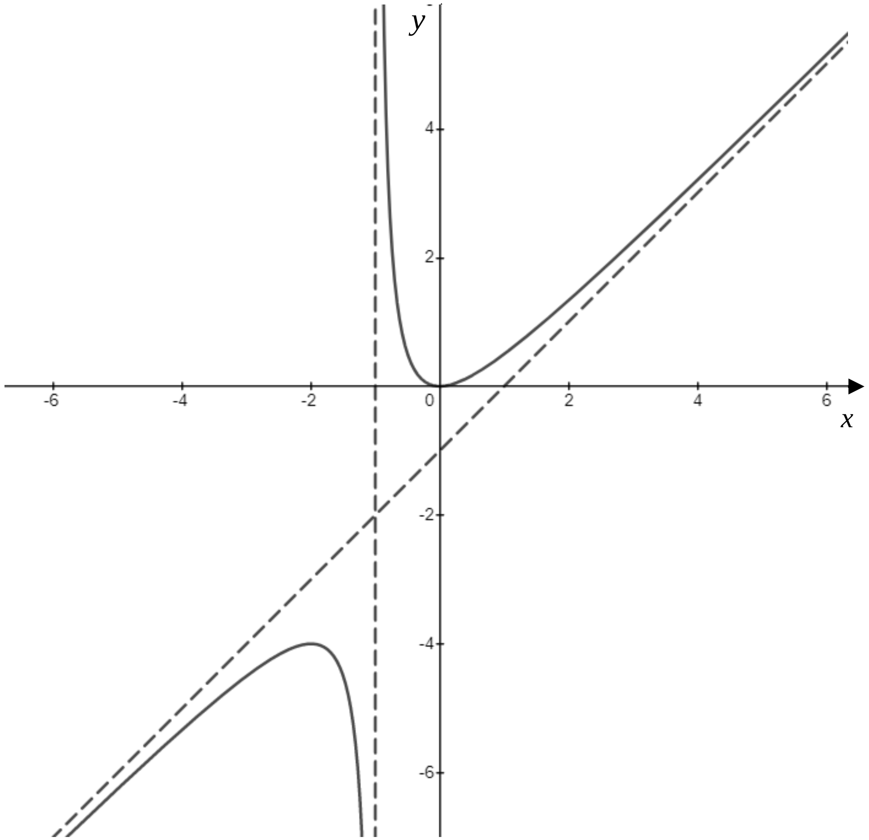
Question	Answer	Marks	Guidance
5(a)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -3 \\ 3 & 0 & -1 \end{vmatrix} = \begin{pmatrix} 2 \\ -8 \\ 6 \end{pmatrix} \sim \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}$	M1 A1	Finds perpendicular to Π_1.
	$1(1)-4(-1)+3(-2)=-1$	M1	Uses point on Π_1.
	$x-4y+3z=-1$	A1	
		4	
5(b)	$\begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}=1-4+3=0$	M1 A1	Shows dot product with direction of line is 0.
		2	
5(c)	$\frac{1}{\sqrt{1^2+4^2+3^2}}\begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}\bullet\begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}$ or $\frac{1}{\sqrt{1^2+4^2+3^2}}\left(\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}\bullet\begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}+1\right)$	M1 A1	Uses correct formula for distance from point on l to Π_1. $\frac{1}{\sqrt{1^2+4^2+3^2}}(-3.1+0.-4+1.3+1)$
	$\frac{1}{\sqrt{26}}(=0.196)$	A1	
		3	

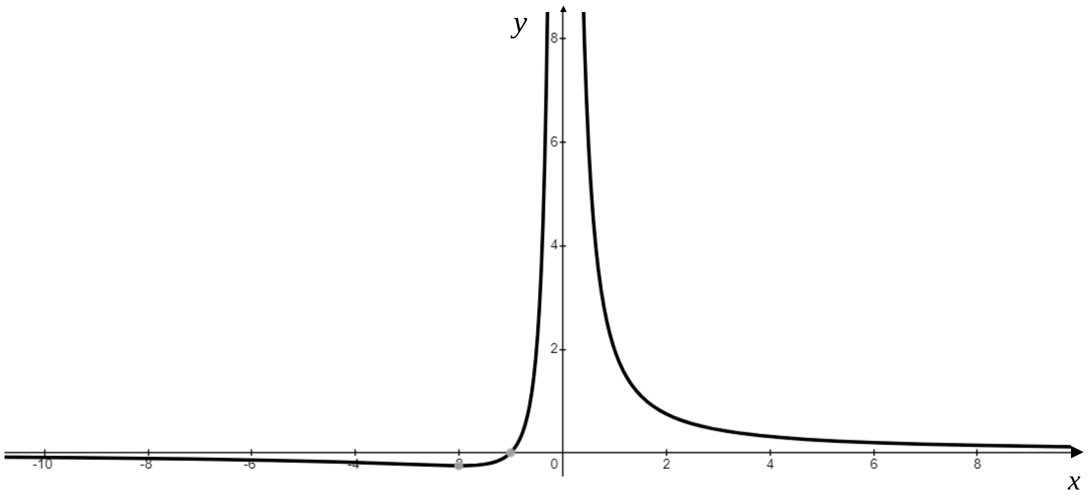
Question	Answer	Marks	Guidance
5(d)	States point common to both planes e.g. $\begin{pmatrix} \frac{1}{15} \\ \frac{4}{15} \\ 0 \end{pmatrix}$.	B1	$\begin{pmatrix} \frac{5}{7} \\ 0 \\ \frac{-4}{7} \end{pmatrix}$ or $\begin{pmatrix} 0 \\ \frac{5}{17} \\ \frac{1}{17} \end{pmatrix}$ or alternative.
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 3 \\ 3 & 3 & 2 \end{vmatrix} = \begin{pmatrix} -17 \\ 7 \\ 15 \end{pmatrix}$	M1 A1	Finds direction of line.
	$\mathbf{r} = \begin{pmatrix} \frac{5}{7} \\ 0 \\ -\frac{4}{7} \end{pmatrix} + \lambda \begin{pmatrix} -17 \\ 7 \\ 15 \end{pmatrix}$	A1	OE.
		4	

Question	Answer	Marks	Guidance
6(a)	$\theta = 0$ 	B1	Initial line drawn. Correct shape, r strictly decreasing.
		B1	Correct shape at extremities.
	$1 - e^{-\frac{1}{2}\pi}$	B1	May be seen on *their* diagram.
		3	
6(b)	$\frac{1}{2}\int_0^{\frac{1}{2}\pi}\left(e^{-\theta} - e^{-\frac{1}{2}\pi}\right)^2 d\theta$	M1	Uses correct formula with correct limits.
	$\frac{1}{2}\int_0^{\frac{1}{2}\pi} e^{-2\theta} - 2e^{-\theta-\frac{1}{2}\pi} + e^{-\pi}\, d\theta$	A1	
	$\frac{1}{2}\left[-\frac{1}{2}e^{-2\theta} + 2e^{-\theta-\frac{1}{2}\pi} + e^{-\pi}\theta\right]_0^{\frac{1}{2}\pi}$	M1 A1	Integrates.
	$\frac{1}{2}\left(-\frac{1}{2}e^{-\pi} + 2e^{-\pi} + \frac{1}{2}\pi e^{-\pi} + \frac{1}{2} - 2e^{-\frac{1}{2}\pi}\right) = \frac{3}{4}e^{-\pi} + \frac{1}{4}\pi e^{-\pi} - e^{-\frac{1}{2}\pi} + \frac{1}{4}$	A1	
		5	

Question	Answer	Marks	Guidance
6(c)	$y=\left(e^{-\theta}-e^{-\frac{1}{2}\pi}\right)\sin\theta$	B1	Uses $y=r\sin\theta$
	$\frac{dy}{d\theta}=\left(e^{-\theta}-e^{-\frac{1}{2}\pi}\right)\cos\theta+\sin\theta\left(-e^{-\theta}\right)=0$	M1 A1	Sets derivative equal to zero.
	$\left[\theta\neq\frac{1}{2}\pi\Rightarrow\right]1+\left(\frac{-e^{-\theta}}{e^{-\theta}-e^{-\frac{1}{2}\pi}}\right)\tan\theta=0\Rightarrow1-e^{\theta-\frac{1}{2}\pi}-\tan\theta=0$	A1	AG.
	$1-e^{0.56-\frac{1}{2}\pi}-\tan0.56=0.00912$ and $1-e^{0.57-\frac{1}{2}\pi}-\tan0.57=-0.00856$	B1	Shows sign change.
		5	

Question	Answer	Marks	Guidance
7(a)	$x=-1$	B1	Vertical asymptote.
	$y=\frac{(x+1)(x-1)+1}{x+1}$	M1	Oblique asymptote.
	$y=x-1$	A1	
		3	
7(b)	$\frac{dy}{dx}=\frac{x^2+2x}{(x+1)^2}=0$	M1	Sets $\frac{dy}{dx}=0$.
	$(0,0),\ (-2,-4)$	A1	
		2	

Question	Answer	Marks	Guidance
7(c)		B1	Axes and asymptotes.
		B1	Left branch correct.
		B1	Right branch correct.
		3	
7(d)	$\left(-2,-\frac{1}{4}\right)$	B1 B1	B1 for each correct coordinate. SC B1 for $\left(-2,-\frac{1}{4}\right)$ and $(0,0)$.
		2	

Question	Answer	Marks	Guidance
7(e)		B1	Left branch correct.
		B1	Right branch correct.
	$\dfrac{x^2}{x+1}=1$ or $\dfrac{x^2}{x+1}=-1$ $x^2-x-1=0$	M2	Finds critical points, award M1 for each case.
	$x=\frac{1}{2}-\frac{1}{2}\sqrt{5}$ or $x=\frac{1}{2}+\frac{1}{2}\sqrt{5}$	A1	
	$x<-1,\ \frac{1}{2}-\frac{1}{2}\sqrt{5}<x<\frac{1}{2}+\frac{1}{2}\sqrt{5},\ \ x\neq 0$	B1	Condone missing $x\neq 0$.
		6	

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CANDIDATE NAME	
CENTRE NUMBER	
CANDIDATE NUMBER	

1842585985

FURTHER MATHEMATICS **9231/11**

Paper 1 Further Pure Mathematics 1 **May/June 2023**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **20** pages. Any blank pages are indicated.

DC (LO) 327445

1 Let $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$.

(a) Prove by mathematical induction that, for all positive integers n,

$$2\mathbf{A}^n = \begin{pmatrix} 2 \times 3^n & 0 \\ 3^n - 1 & 2 \end{pmatrix}.$$

[5]

(b) Find, in terms of n, the inverse of $\mathbf{A}^n$. [2]

2 The cubic equation $x^3 + 4x^2 + 6x + 1 = 0$ has roots α, β, γ.

(a) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [2]

(b) Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{n}\left((\alpha+r)^2 + (\beta+r)^2 + (\gamma+r)^2\right) = n(n^2 + an + b),$$

where a and b are constants to be determined. [6]

3 (a) Use the method of differences to find $\sum_{r=1}^{n} \frac{1}{(kr+1)(kr-k+1)}$ in terms of n and k, where k is a positive constant. [4]

(b) Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{(kr+1)(kr-k+1)}$. [1]

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(c) Find also $\sum_{r=n}^{n^2} \frac{1}{(kr+1)(kr-k+1)}$ in terms of n and k. [2]

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4 The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} a & b^2 \\ c^2 & a \end{pmatrix}$, where a, b, c are real constants and $b \neq 0$.

(a) Show that **M** does not represent a rotation about the origin. [2]

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(b) Find the equations of the invariant lines, through the origin, of the transformation in the $x-y$ plane represented by **M**. [5]

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It is given that **M** represents the sequence of two transformations in the $x-y$ plane given by an enlargement, centre the origin, scale factor 5 followed by a shear, x-axis fixed, with $(0, 1)$ mapped to $(5, 1)$.

(c) Find **M**. [3]

(d) The triangle DEF in the $x-y$ plane is transformed by **M** onto triangle PQR.

Given that the area of triangle DEF is $12\ \text{cm}^2$, find the area of triangle PQR. [2]

5 The curve C has polar equation $r^2 = \frac{1}{\theta^2 + 1}$, for $0 \leqslant \theta \leqslant \pi$.

(a) Sketch C and state the polar coordinates of the point of C furthest from the pole. [3]

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(b) Find the area of the region enclosed by C, the initial line, and the half-line $\theta = \pi$. [4]

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(c) Show that, at the point of C furthest from the initial line,

$$\left(\theta + \frac{1}{\theta}\right)\cot\theta - 1 = 0$$

and verify that this equation has a root between 1.1 and 1.2. [5]

6 The curve C has equation $y = \dfrac{x^2 + 2x - 15}{x - 2}$.

(a) Find the equations of the asymptotes of C. [3]

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(b) Show that C has no stationary points. [3]

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(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

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(d) Sketch the curve with equation $y = \left|\dfrac{x^2 - 2x - 15}{x - 2}\right|$. [2]

(e) Find the set of values of x for which $\left|\dfrac{2x^2+4x-30}{x-2}\right| < 15$. [4]

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7 The plane Π_1 has equation $r = -4\mathbf{j} - 3\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k}) + \mu(\mathbf{i} + \mathbf{j} - \mathbf{k})$.

(a) Obtain an equation of Π_1 in the form $px + qy + rz = d$. [4]

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(b) The plane Π_2 has equation $\mathbf{r}.(-5\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) = 4$.

Find a vector equation of the line of intersection of Π_1 and Π_2. [4]

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The line l passes through the point A with position vector $a\mathbf{i}+a\mathbf{j}+(a-7)\mathbf{k}$ and is parallel to $(1-b)\mathbf{i}+b\mathbf{j}+b\mathbf{k}$, where a and b are positive constants.

(c) Given that the perpendicular distance from A to Π_1 is $\sqrt{2}$, find the value of a. [2]

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(d) Given that the obtuse angle between l and Π_1 is $\frac{3}{4}\pi$, find the exact value of b. [4]

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Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

Cambridge Assessment International Education

Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/11**

Paper 1 Further Pure Mathematics 11 **May/June 2023**

MARK SCHEME

Maximum Mark: 75

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

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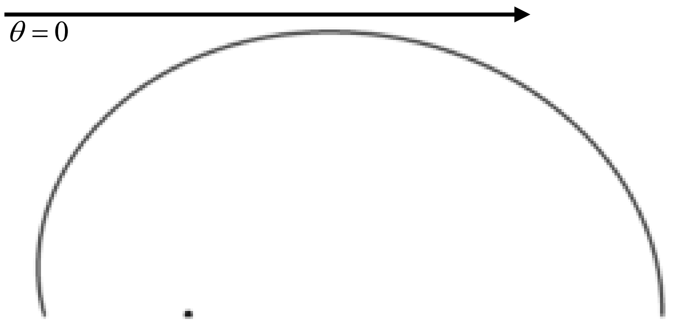
This document consists of **16** printed pages.

Question	Answer	Marks	Guidance
1(a)	$2\mathbf{A} = \begin{pmatrix} 6 & 0 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2\times 3 & 0 \\ 3-1 & 2 \end{pmatrix}$ so true when $n=1$.	B1	States base case.
	Assume that it is true for $n=k$, so $2\mathbf{A}^k = \begin{pmatrix} 2\times 3^k & 0 \\ 3^k-1 & 2 \end{pmatrix}$.	B1	States inductive hypothesis.
	Then $2\mathbf{A}^{k+1} = \begin{pmatrix} 2\times 3^k & 0 \\ 3^k-1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2\times 3^{k+1} & 0 \\ 3^{k+1}-3+2 & 2 \end{pmatrix}$	M1A1	Multiplies $2\mathbf{A}^k$ with $\mathbf{A}$.
	So, it is also true for $n=k+1$. Hence, by induction, true for all positive integers.	A1	States conclusion.
		5	
1(b)	$\det \mathbf{A}^n = \det\begin{pmatrix} 3^n & 0 \\ \frac{1}{2}\left(3^n-1\right) & 1 \end{pmatrix} = 3^n$ Or a multiple of $\mathbf{A}^{-n} = 2^{-1}3^{-n}\begin{pmatrix} 2 & 0 \\ 1-3^n & 2\times 3^n \end{pmatrix}$ seen.	B1	
	$\mathbf{A}^{-n} = 3^{-n}\begin{pmatrix} 1 & 0 \\ \frac{1}{2}\left(1-3^n\right) & 3^n \end{pmatrix}$	B1	OE $\mathbf{A}^{-n} = 2^{-1}3^{-n}\begin{pmatrix} 2 & 0 \\ 1-3^n & 2\times 3^n \end{pmatrix}$
		2	

Question	Answer	Marks	Guidance
2(a)	$(-4)^2 - 2(6)$	M1	Uses formula for sum of squares.
	4	A1	
		2	
2(b)	$(\alpha + r)^2 = \alpha^2 + 2\alpha r + r^2$	B1	Expands.
	$\sum_{r=1}^{n}\left((\alpha + r)^2 + (\beta + r)^2 + (\gamma + r)^2\right) = \sum_{r=1}^{n}\left(4 + 2(-4)r + 3r^2\right)$	M1 A1	Collects like terms and uses $\alpha + \beta + \lambda = -4$ and *their* result from part **(a)**.
	$4n - 4n(n+1) + \frac{1}{2}n(n+1)(2n+1)$	M1	Applies formulae from MF19.
	$-4n^2 + \frac{1}{2}n(n+1)(2n+1)$ $= n\left(-4n + \frac{1}{2}(2n^2 + 3n + 1)\right)$ $= n\left(n^2 - \frac{5}{2}n + \frac{1}{2}\right)$	M1 A1	Simplifies.
		6	

Question	Answer	Marks	Guidance
3(a)	$\dfrac{1}{(kr+1)(kr-k+1)} = \dfrac{1}{k}\left(\dfrac{1}{k(r-1)+1} - \dfrac{1}{kr+1}\right)$	M1 A1	Finds partial fractions.
	$\displaystyle\sum_{r=1}^{n} \dfrac{1}{(kr+1)(kr-k+1)} = \dfrac{1}{k}\left(1 - \dfrac{1}{k+1} + \dfrac{1}{k+1} - \dfrac{1}{2k+1} + \ldots + \dfrac{1}{k(n-1)+1} - \dfrac{1}{kn+1}\right)$	M1	Writes at least three complete terms, including last.
	$\dfrac{1}{k}\left(1 - \dfrac{1}{kn+1}\right)$	A1	OE e.g. $\dfrac{n}{kn+1}$
		4	
3(b)	$\dfrac{1}{k}$	B1	
		1	
3(c)	$\displaystyle\sum_{r=n}^{n^2} \dfrac{1}{(kr+1)(kr-k+1)} = \sum_{r=1}^{n^2} \dfrac{1}{(kr+1)(kr-k+1)} - \sum_{r=1}^{n-1} \dfrac{1}{(kr+1)(kr-k+1)}$	M1	Or applies the method of differences again.
	$\dfrac{1}{k}\left(1 - \dfrac{1}{kn^2+1} - \left(1 - \dfrac{1}{k(n-1)+1}\right)\right) = \dfrac{1}{k}\left(\dfrac{1}{k(n-1)+1} - \dfrac{1}{kn^2+1}\right)$	A1	OE e.g. $\dfrac{n^2}{kn^2+1} - \dfrac{(n-1)}{k(n-1)+1}$
		2	

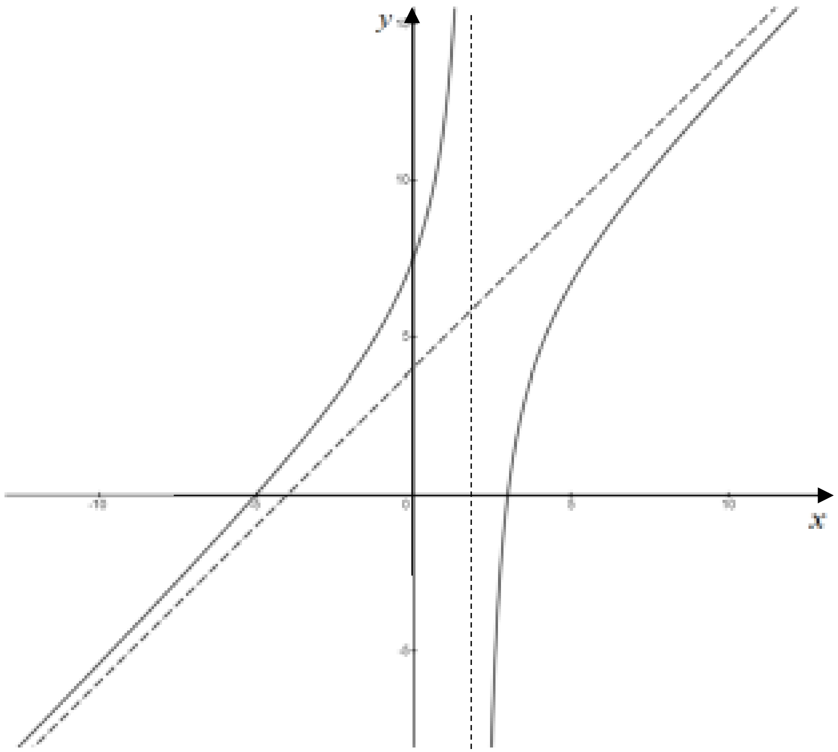
Question	Answer	Marks	Guidance
4(a)	$\begin{pmatrix} a & b^2 \\ c^2 & a \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$	M1	Uses correct matrix for rotation.
	$b^2 = -c^2$ which is impossible since b and c are real and $b \neq 0$.	A1	AG
		2	
4(b)	$\begin{pmatrix} a & b^2 \\ c^2 & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + b^2 y \\ c^2 x + ay \end{pmatrix}$	B1	Transforms $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} X \\ Y \end{pmatrix}$.
	$c^2 x + amx = m\left(ax + b^2 mx\right)$	M1 A1	Uses $y = mx$ and $Y = mX$.
	$c^2 + am = ma + b^2 m^2 \Rightarrow c^2 = b^2 m^2$	A1	
	$y = \frac{c}{b}x$ and $y = -\frac{c}{b}x$	A1	
		5	
4(c)	$\mathbf{M} = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$	M1 A1*	Award M1 if matrices correct but order is wrong.
	$\begin{pmatrix} 5 & 5^2 \\ 0 & 5 \end{pmatrix}$	DA1	Dep: previous A1
		3	
4(d)	$12 \times \det \mathbf{M}$	M1	Using *their* **M**.
	$300\ \text{cm}^2$	A1FT	
		2	

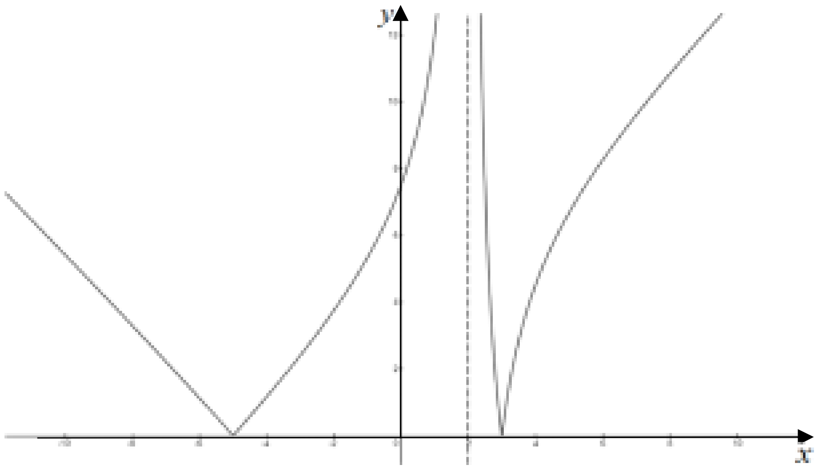
Question	Answer	Marks	Guidance
5(a)		B1*	Correct shape and domain, polar graph with r strictly decreasing but condone if not strictly decreasing close to $\theta = \pi$.
		DB1	Fully correct including shape at $\theta = 0$ correct shape at $\theta = \pi$ correct
	$(1, 0)$	B1	Identified as point furthest from the pole and given as coordinates.
		3	
5(b)	$\frac{1}{2}\int_0^{\pi} \frac{1}{\theta^2 + 1}\, d\theta$	M1	Uses correct formula with correct limits.
	$\frac{1}{2}\left[\tan^{-1}\theta\right]_0^{\pi}$	M1 A1	Integrates $\frac{1}{\theta^2 + 1}$.
	$\frac{1}{2}\tan^{-1}\pi = 0.631$	A1	
		4	

Question	Answer	Marks	Guidance
5(c)	$y = \dfrac{\sin\theta}{\sqrt{\theta^2+1}}$	B1	Uses $y = r\sin\theta$
	$\dfrac{dy}{d\theta} = \dfrac{\left(\theta^2+1\right)^{\frac{1}{2}}\cos\theta - \theta\left(\theta^2+1\right)^{-\frac{1}{2}}\sin\theta}{\theta^2+1} = 0$	M1 A1	Sets derivative equal to zero.
	$\theta \neq 0 \Rightarrow \left(\theta + \dfrac{1}{\theta}\right)\cot\theta - 1 = 0$	A1	AG.
	$\left(1.1+\dfrac{1}{1.1}\right)\cot 1.1 - 1 = 0.02\ldots$ and $\left(1.2+\dfrac{1}{1.2}\right)\cot 1.2 - 1 = -0.209\ldots$	B1	Shows sign change (1sf or better).
		5	

Question	Answer	Marks	Guidance
6(a)	$x = 2$	B1	States vertical asymptote.
	$x^2 + 2x - 15 = (x-2)(x+4) - 7 \Rightarrow y = x + 4$	M1 A1	Finds oblique asymptote.
		3	

Question	Answer	Marks	Guidance
6(b)	$\frac{dy}{dx} = \frac{(x-2)(2x+2)-(x^2+2x-15)}{(x-2)^2}$	M1	Differentiates.
	$x^2 - 4x + 11 = 0$ $\left(\text{or } \frac{dy}{dx} = 1 + \frac{7}{(x-2)^2}\right)$	A1	Forms quadratic equation or simplifies $\frac{dy}{dx}$.
	$4^2 - 4(1)(11) = -28 < 0$ (or $y' > 0$) There are no turning points.	A1	Correct conclusion.
		3	

Question	Answer	Marks	Guidance
6(c)		B1	Axes and asymptotes.
		B1	Branches correct.
	$(0,7.5),(-5,0),(3,0)$	B1	States coordinates of intersections with axes.
		3	

Question	Answer	Marks	Guidance
6(d)		B1FT	FT from sketch in **(c)**.
		B1	Correct shape at infinity and on x axis.
		2	
6(e)	$\frac{x^2+2x-15}{x-2}=\frac{15}{2}$ or $\frac{x^2+2x-15}{x-2}=-\frac{15}{2}$ $x^2-\frac{11}{2}x=0$ or $x^2+\frac{19}{2}x-30=0$	M2	Finds critical points, award M1 for each case.
	$x=0,\frac{11}{2}$ or $x=-12,\frac{5}{2}$	A1	
	$-12<x<0$ or $\frac{5}{2}<x<\frac{11}{2}$	A1FT	
		4	

Question	Answer	Marks	Guidance
7(a)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$	M1 A1	Finds common perpendicular.
	$(-4)+(-3)=-7 \Rightarrow y+z=-7$	M1 A1	Substitutes point.
		4	
7(b)	States point common to both planes e.g. $\begin{pmatrix} -7 \\ -2 \\ -5 \end{pmatrix}$.	B1 FT	
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ -5 & 3 & 5 \end{vmatrix} = \begin{pmatrix} 2 \\ -5 \\ 5 \end{pmatrix}$	M1 A1FT	Finds direction of line.
	$\mathbf{r} = \begin{pmatrix} -7 \\ -2 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -5 \\ 5 \end{pmatrix}$	A1	OE.
		4	
7(c)	$\left\lvert \frac{a+a-7+7}{\sqrt{2}} \right\rvert = \sqrt{2}$	M1	Uses correct formula for distance from A to Π_1.
	$a=1$	A1	
		2	

Question	Answer	Marks	Guidance
7(d)	$\left\lvert \dfrac{b+b}{\sqrt{2}\sqrt{(1-b)^2+2b^2}} \right\rvert = \frac{1}{2}\sqrt{2}$	M1 A1	Uses correct formula.
	$2b = \sqrt{(1-b)^2+2b^2} \Rightarrow b^2+2b-1=0$	M1	Solves for b.
	$b=-1+\sqrt{2}$	A1	CAO
		4	

Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

0834557711

FURTHER MATHEMATICS **9231/13**

Paper 1 Further Pure Mathematics 1 **May/June 2023**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Any blank pages are indicated.

DC (CE) 311426/2

1 Prove by mathematical induction that, for all positive integers n, $5^{3n} + 32^n - 33$ is divisible by 31. [6]

2 **(a)** Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{n}(6r^2+6r-5) = an^3+bn^2+cn,$$

where a, b and c are integers to be determined. [2]

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(b) Use the method of differences to find $\sum_{r=1}^{n}\frac{6r^2+6r-5}{r^2+r}$ in terms of n. [4]

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(c) Find also $\sum_{r=n+1}^{2n} \frac{6r^2+6r-5}{r^2+r}$ in terms of n. [2]

3 The equation $x^4 - x^2 + 2x + 5 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\alpha^2, \beta^2, \gamma^2, \delta^2$ and state the value of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$. [4]

(b) Find the value of $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{1}{\delta^2}$. [3]

(c) Find the value of $\alpha^4+\beta^4+\gamma^4+\delta^4$. [2]

4 The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, where $0 < \theta < \pi$ and k is a non-zero constant.

The matrix $\mathbf{M}$ represents a sequence of two geometrical transformations, one of which is a shear.

(a) Describe fully the other transformation and state the order in which the transformations are applied. [3]

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(b) Write $\mathbf{M}^{-1}$ as the product of two matrices, neither of which is $\mathbf{I}$. [2]

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(c) Find, in terms of k, the value of $\tan\theta$ for which $\mathbf{M} - \mathbf{I}$ is singular. [5]

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(d) Given that $k = 2\sqrt{3}$ and $\theta = \frac{1}{3}\pi$, show that the invariant points of the transformation represented by **M** lie on the line $3y + \sqrt{3}x = 0$. [4]

5 **(a)** Show that the curve with Cartesian equation

$$x^2 - y^2 = a,$$

where a is a positive constant, has polar equation $r^2 = a\sec 2\theta$. [3]

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The curve C has polar equation $r^2 = a\sec 2\theta$, where a is a positive constant, for $0 \leqslant \theta < \frac{1}{4}\pi$.

(b) Sketch C and state the minimum distance of C from the pole. [3]

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(c) Find, in terms of a, the exact value of the area of the region enclosed by C, the initial line, and the half-line $\theta = \frac{1}{12}\pi$. [You may use any result from the list of formulae (MF19) without proof.] [4]

6 The points A, B, C have position vectors

$$\mathbf{i}+\mathbf{j}, \quad -\mathbf{i}+2\mathbf{j}+4\mathbf{k}, \quad -2\mathbf{i}+\mathbf{j}+3\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

(b) Find the perpendicular distance from O to the plane ABC. [2]

(c) Find a vector equation of the common perpendicular to the lines OC and AB. [8]

7 The curve C has equation $y = \dfrac{x^2 + 2x + 1}{x - 3}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Find the coordinates of the turning points on C. [3]

(c) Sketch C. [3]

(d) Sketch the curves with equations $y = \left|\frac{x^2+2x+1}{x-3}\right|$ and $y^2 = \frac{x^2+2x+1}{x-3}$ on a single diagram, clearly identifying each curve. [4]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

Cambridge Assessment International Education

Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/13**

Paper 1 Further Pure Mathematics 13 **May/June 2023**

MARK SCHEME

Maximum Mark: 75

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

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Cambridge International is publishing the mark schemes for the May/June 2023 series for most Cambridge IGCSE, Cambridge International A and AS Level and Cambridge Pre-U components, and some Cambridge O Level components.

Question	Answer	Marks	Guidance
1	$5^3 + 32 - 33 = 124$ is divisible by 31.	B1	Checks base case.
	Assume that $5^{3k} + 32^k - 33$ is divisible by 31 for some positive integer k.	B1	States inductive hypothesis.
	$5^{3k+3} + 32^{k+1} - 33 = (124+1)5^{3k} + (31+1)32^k - 33$	M1 A1	Separates $5^{3k} + 32^k - 33$ or considers difference.
	is divisible by 31 because $124 \times 5^{3k} + 31 \times 32^k$ is divisible by 31.	A1	
	Hence, by induction, true for every positive integer n.	A1	
		6	

Question	Answer	Marks	Guidance
2(a)	$6\left(\frac{1}{6}n(n+1)(2n+1)\right) + 6\left(\frac{1}{2}n(n+1)\right)[-5n]$	M1	Substitutes formulae for $\sum r^2$ and $\sum r$.
	$2n^3 + 6n^2 - n$	A1	
		2	

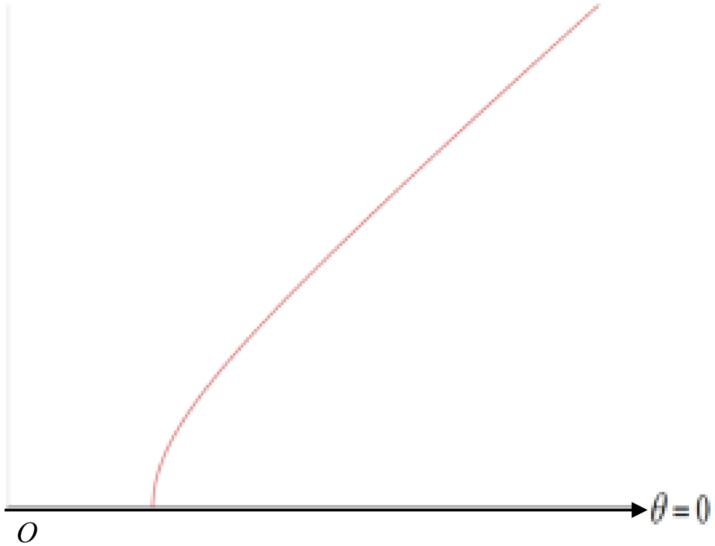
Question	Answer	Marks	Guidance
2(b)	$\frac{6r^2+6r-5}{r^2+r}=6-\frac{5}{r(r+1)}$	**B1**	Divides by denominator.
	$\frac{1}{r(r+1)}=\frac{1}{r}-\frac{1}{r+1}$	**B1**	Finds partial fractions.
	$\sum_{r=1}^{n}\frac{1}{r(r+1)}=1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\ldots+\frac{1}{n}-\frac{1}{n+1}$	**M1**	Writes at least three complete terms, including the last term, to show cancelation.
	$\sum_{r=1}^{n}\frac{6r^2+6r-5}{r^2+r}=6n-5+\frac{5}{n+1}$	**A1**	OE e.g. $\frac{6n^2+n}{n+1}$
		4	
2(c)	$\sum_{r=1}^{2n}\frac{6r^2+6r-5}{r^2+r}-\sum_{r=1}^{n}\frac{6r^2+6r-5}{r^2+r}$ $=12n-5+\frac{5}{2n+1}-6n+5-\frac{5}{n+1}$	**M1**	Or uses method of differences again.
	$12n-5+\frac{5}{2n+1}-6n+5-\frac{5}{n+1}=6n+\frac{5}{2n+1}-\frac{5}{n+1}$	**A1**	Or $6n-\frac{5n}{(n+1)(2n+1)}$. OE, like terms collected.
		2	

Question	Answer	Marks	Guidance
3(a)	$y = x^2$ so, $x = y^{\frac{1}{2}}$	B1	Correct substitution.
	$y^2 - y + 2y^{\frac{1}{2}} + 5 = 0$ so, $\left(2y^{\frac{1}{2}}\right)^2 = \left(-y^2 + y - 5\right)^2$	M1	Obtains an equation which eliminates radicals.
	$y^4 - 2y^3 + 11y^2 - 14y + 25 = 0$	A1	
	$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 2$	B1FT	
		4	
3(b)	$\alpha^2\beta^2\gamma^2\delta^2 = 25$	B1FT	
	$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2} = \frac{\alpha^2\beta^2\delta^2 + \alpha^2\beta^2\gamma^2 + \beta^2\gamma^2\delta^2 + \alpha^2\gamma^2\delta^2}{\alpha^2\beta^2\gamma^2\delta^2}$	M1	Relates to coefficients.
	$\frac{14}{25}$	A1	CAO
		3	
3(c)	$\alpha^4 + \beta^4 + \gamma^4 + \delta^4 = \left(\alpha^2 + \beta^2 + \gamma^2 + \delta^2\right)^2 - 2\left(\alpha^2\beta^2 + \alpha^2\gamma^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2 + \gamma^2\delta^2\right)$ $= 2^2 - 2(11)$	M1	Uses formula for sum of squares or uses original equation.
	-18	A1	
		2	

Question	Answer	Marks	Guidance
4(a)	Rotation [anticlockwise]	**B1**	
	about the origin through angle 2θ .	**B1**	
	Shear [in the x-direction] followed by a rotation [anticlockwise about the origin through angle 2θ].	**B1**	
		3	
4(b)	$\begin{pmatrix}1 & k\\0 & 1\end{pmatrix}^{-1}=\begin{pmatrix}1 & -k\\0 & 1\end{pmatrix}$, $\begin{pmatrix}\cos 2\theta & -\sin 2\theta\\\sin 2\theta & \cos 2\theta\end{pmatrix}^{-1}=\begin{pmatrix}\cos 2\theta & \sin 2\theta\\-\sin 2\theta & \cos 2\theta\end{pmatrix}$	**B1**	
	$\mathbf{M}^{-1}=\begin{pmatrix}1 & -k\\0 & 1\end{pmatrix}\begin{pmatrix}\cos 2\theta & \sin 2\theta\\-\sin 2\theta & \cos 2\theta\end{pmatrix}$	**B1**	Correct order
		2	
4(c)	$\mathbf{M}-\mathbf{I}=\begin{pmatrix}\cos 2\theta-1 & k\cos 2\theta-\sin 2\theta\\\sin 2\theta & k\sin 2\theta+\cos 2\theta-1\end{pmatrix}$	**B1**	
	$(\cos 2\theta-1)(k\sin 2\theta+\cos 2\theta-1)-k\sin 2\theta\cos 2\theta+\sin^2 2\theta[=0]$	**M1**	Evaluates $\det(\mathbf{M}-\mathbf{I})$
	$2-2\cos 2\theta-k\sin 2\theta=0$	**A1**	Brackets removed correctly and = 0
	$4\sin^2\theta=2k\sin\theta\cos\theta$	**M1**	Uses $1-\cos 2\theta=2\sin^2\theta$ and $\sin 2\theta=2\sin\theta\cos\theta$ or all necessary double angle formulae.
	$\tan\theta=\frac{1}{2}k$	**A1**	
		5	

Question	Answer	Marks	Guidance
4(d)	$\mathbf{M}=\begin{pmatrix}-\frac{1}{2} & -\frac{3}{2}\sqrt{3}\\ \frac{1}{2}\sqrt{3} & \frac{5}{2}\end{pmatrix}$.	B1	
	$\begin{pmatrix}-\frac{1}{2} & -\frac{3}{2}\sqrt{3}\\ \frac{1}{2}\sqrt{3} & \frac{5}{2}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}x-\frac{3}{2}\sqrt{3}y\\ \frac{1}{2}\sqrt{3}x+\frac{5}{2}y\end{pmatrix}$	B1FT	Transforms $\begin{pmatrix}x\\ y\end{pmatrix}$ to $\begin{pmatrix}X\\ Y\end{pmatrix}$
	$-\frac{1}{2}x-\frac{3}{2}\sqrt{3}y=x\left[\Rightarrow -\frac{3}{2}x-\frac{3}{2}\sqrt{3}y=0\Rightarrow x+\sqrt{3}y=0\right]$ and $\frac{1}{2}\sqrt{3}x+\frac{5}{2}y=y\left[\Rightarrow \frac{1}{2}\sqrt{3}x+\frac{3}{2}y=0\Rightarrow \sqrt{3}x+3y=0\right]$	M1	Sets $\begin{pmatrix}X\\ Y\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$
	$\sqrt{3}x+3y=0$	A1	AG.
		4	

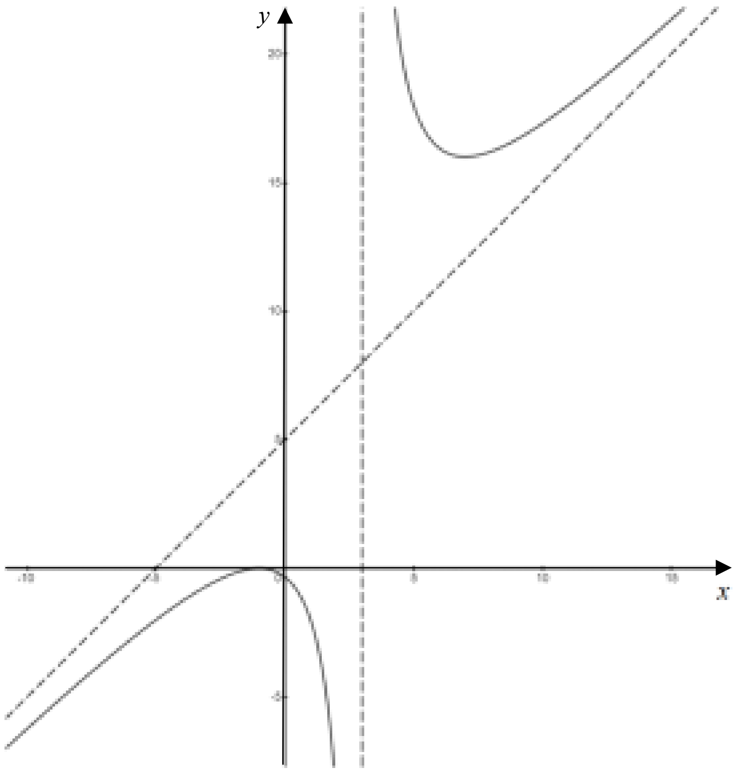
Question	Answer	Marks	Guidance
5(a)	$r^2\left(\cos^2\theta-\sin^2\theta\right)=a$	B1	Uses $x=r\cos\theta$ and/or $y=r\sin\theta$.
	$r^2\cos 2\theta=a$	M1	Applies relevant double angle formulae.
	$r^2=a\sec 2\theta$	A1	AG.
		3	

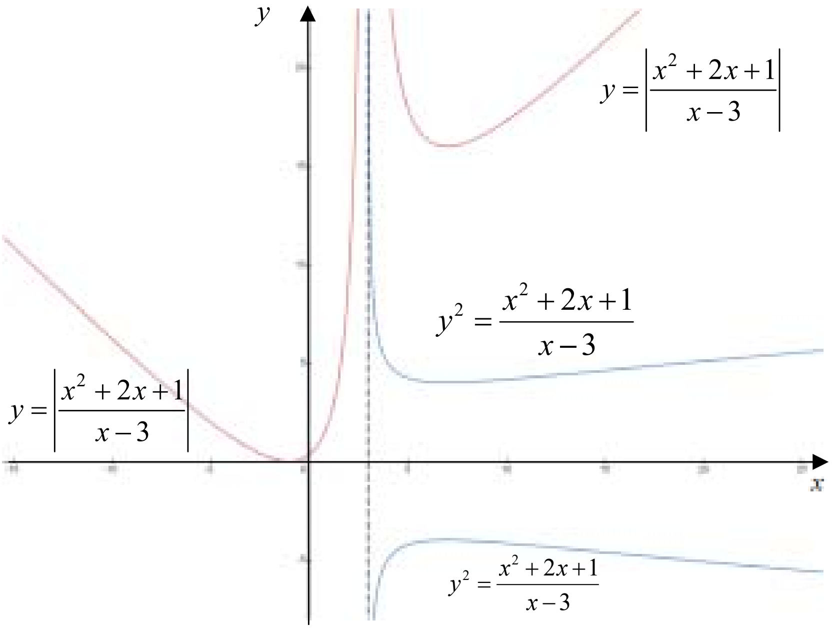
Question	Answer	Marks	Guidance
5(b)		B1*	Initial line drawn. Correct domain and position, r strictly increasing.
		dB1	Also sloping to right, concave on opposite side to pole, correct gradient when $\theta = 0$ and $\theta \to \pi/4$.
	$\sqrt{a}$	B1	
		3	
5(c)	$\frac{1}{2}a\int_0^{\frac{1}{12}\pi} \sec 2\theta \, d\theta$	M1	Uses $\frac{1}{2}\int r^2 \, d\theta$ with correct limits.
	$\frac{1}{4}a\left[\ln\tan\left(\theta + \frac{1}{4}\pi\right)\right]_0^{\frac{1}{12}\pi}$ or $\frac{1}{4}a\left[\ln\left(\tan 2\theta + \sec 2\theta\right)\right]_0^{\frac{1}{12}\pi}$	M1 A1	Integrates.
	$\frac{1}{4}a\ln\sqrt{3} = \frac{1}{8}a\ln 3$	A1	
		4	

Question	Answer	Marks	Guidance
6(a)	$\overrightarrow{AB} = -2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ $\overrightarrow{AC} = -3\mathbf{i} + 3\mathbf{k}$ $\overrightarrow{BC} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$	B1	Finds direction vectors of **two** lines in the plane.
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 4 \\ -3 & 0 & 3 \end{vmatrix} = \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix} \sim \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$	M1 A1FT	Finds normal to the plane *ABC*.
	$1(1) - 2(1) + 1(0) = -1 \Rightarrow x - 2y + z = -1$	M1 A1	Substitutes point. CAO
		5	
6(b)	$\dfrac{1}{\sqrt{1^2 + 2^2 + 1^2}} = \dfrac{1}{\sqrt{6}} = 0.408$	M1 A1FT	Divides by magnitude of normal vector. FT *their* **(a)**.
		2	

Question	Answer	Marks	Guidance
6(c)	$\overrightarrow{OP}=\begin{pmatrix}-2\lambda\\ \lambda\\ 3\lambda\end{pmatrix}, \overrightarrow{OQ}=\begin{pmatrix}1-2\mu\\ 1+\mu\\ 4\mu\end{pmatrix}\Rightarrow\overrightarrow{PQ}=\begin{pmatrix}1-2\mu+2\lambda\\ 1+\mu-\lambda\\ 4\mu-3\lambda\end{pmatrix}$	**M1 A1**	Finds $\overrightarrow{PQ}$, where *P* is a point on *OC* and *Q* is a point on *AB*.
	$\begin{pmatrix}1-2\mu+2\lambda\\ 1+\mu-\lambda\\ 4\mu-3\lambda\end{pmatrix}\cdot\begin{pmatrix}-2\\ 1\\ 3\end{pmatrix}=0$	**M1***	Uses that dot product of $\overrightarrow{PQ}$ with line direction is zero.
	$17\mu-14\lambda=1$	**dM1**	Deduces one equation.
	$\begin{pmatrix}1-2\mu+2\lambda\\ 1+\mu-\lambda\\ 4\mu-3\lambda\end{pmatrix}\cdot\begin{pmatrix}-2\\ 1\\ 4\end{pmatrix}=0\Rightarrow 21\mu-17\lambda=1$	**dM1**	Deduces second equation.
	$\lambda=-\frac{4}{5}\Rightarrow\overrightarrow{OP}=-\frac{4}{5}\begin{pmatrix}-2\\ 1\\ 3\end{pmatrix}$	**dM1 A1**	Solves for λ or μ and substitutes into $\overrightarrow{OP}$.
	$\mathbf{r}=-\frac{4}{5}\begin{pmatrix}-2\\ 1\\ 3\end{pmatrix}+k\begin{pmatrix}1\\ 2\\ 0\end{pmatrix}$	**A1**	OE
		8	

Question	Answer	Marks	Guidance
7(a)	$x=3$	**B1**	States vertical asymptote.
	$y=\frac{(x-3)(x+5)+16}{x-3}=x+5+\frac{16}{x-3}$	**M1**	Finds oblique asymptote.
	$y=x+5$	**A1**	
		3	
7(b)	$\frac{dy}{dx}=1-\frac{16}{(x-3)^2}=0\Rightarrow(x-3)^2=16$	**M1**	Differentiates and sets equal to zero.
	$x=-1,7$	**A1**	Finds x-coordinates
	$(-1,0),(7,16)$	**A1**	States coordinates of turning points.
		3	

Question	Answer	Marks	Guidance
7(c)		B1FT	Axes and labelled asymptotes.
		B1	Upper branch correct.
		B1	Lower branch correct and no additional branches.
		3	

Question	Answer	Marks	Guidance
7(d)		B1FT	Clear labels, axes and their vertical asymptote.
		B1FT	$y=\left\lvert\frac{x^2+2x+1}{x-3}\right\rvert$ correct, FT from their sketch in **(c)**.
		B1	Upper branch of $y^2=\frac{x^2+2x+1}{x-3}$ (positive square root).
		B1FT	Lower branch of $y^2=\frac{x^2+2x+1}{x-3}$ (negative square root). FT from previous mark.
		4	