Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

FURTHER MATHEMATICS **9231/11**

Paper 1 Further Pure Mathematics 1 **October/November 2022**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Any blank pages are indicated.

1 The cubic equation $x^3 + bx^2 + d = 0$ has roots α, β, γ, where $\alpha = \beta$ and $d \neq 0$.

(a) Show that $4b^3 + 27d = 0$. [5]

(b) Given that $2\alpha^2 + \gamma^2 = 3b$, find the values of b and d. [3]

2 Prove by mathematical induction that, for all positive integers n, $7^{2n} + 97^{n} - 50$ is divisible by 48. [6]

3 **(a)** By considering $(2r+1)^3 - (2r-1)^3$, use the method of differences to prove that

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1).$$ [5]

Let $S_n = 1^2 + 3\times2^2 + 3^2 + 3\times4^2 + 5^2 + 3\times6^2 + \ldots + \left(2+(-1)^n\right)n^2$.

(b) Show that $S_{2n} = \frac{1}{3}n(2n+1)(an+b)$, where a and b are integers to be determined. [3]

(c) State the value of $\lim_{n\to\infty} \frac{S_{2n}}{n^3}$. [1]

4 The plane Π contains the lines $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} + \lambda(-\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ and $\mathbf{r} = 4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + \mu(3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$.

(a) Find a Cartesian equation of Π, giving your answer in the form $ax + by + cz = d$. [4]

The line l passes through the point P with position vector $2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and is parallel to the vector $\mathbf{j} + \mathbf{k}$.

(b) Find the acute angle between l and Π. [3]

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(c) Find the position vector of the foot of the perpendicular from P to Π. [4]

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5 The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, where k is a constant.

(a) The matrix **M** represents a sequence of two geometrical transformations.

State the type of each transformation, and make clear the order in which they are applied. [2]

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(b) The triangle ABC in the x–y plane is transformed by **M** onto triangle DEF.

Find, in terms of k, the single matrix which transforms triangle DEF onto triangle ABC. [2]

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(c) Find the set of values of k for which the transformation represented by **M** has no invariant lines through the origin. [7]

6 **(a)** Show that the curve with Cartesian equation

$$\left(x^2+y^2\right)^2 = 36\left(x^2-y^2\right)$$

has polar equation $r^2 = 36\cos 2\theta$. [3]

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The curve C has polar equation $r^2 = 36\cos 2\theta$, for $-\frac{1}{4}\pi \leqslant \theta \leqslant \frac{1}{4}\pi$.

(b) Sketch C and state the maximum distance of a point on C from the pole. [3]

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(c) Find the area of the region enclosed by C. [2]

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(d) Find the maximum distance of a point on C from the initial line, giving the answer in exact form. [6]

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7 The curve C has equation $y = \dfrac{5x^2}{5x-2}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Find the coordinates of the stationary points on C. [4]

(c) Sketch C. [3]

(d) Sketch the curve with equation $y = \left|\frac{5x^2}{5x-2}\right|$ and find in exact form the set of values of x for which $\left|\frac{5x^2}{5x-2}\right| < 2$. [6]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

Cambridge Assessment International Education

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FURTHER MATHEMATICS **9231/11**

Paper 1 Further Pure Mathematics 1 **October/November 2022**

MARK SCHEME

Maximum Mark: 75

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the October/November 2022 series for most Cambridge IGCSE™, Cambridge International A and AS Level components and some Cambridge O Level components.

Question	Answer	Marks	Guidance
1(a)	$2\alpha+\gamma=-b$ $\alpha^2+2\alpha\gamma=0$	**B1**	
	$\alpha=-2\gamma$ leading to $-4\gamma+\gamma=-b$	**M1**	Solves simultaneous equations, or, express b and d in terms of α and γ.
	$\gamma=\frac{1}{3}b,\ \alpha=-\frac{2}{3}b$	**A1**	$b=3\gamma=-\frac{3}{2}\alpha$ and $d=-\alpha^2\gamma$.
	$\alpha^2\gamma=-d$ leading to $\frac{4}{27}b^3=-d$ leading to $4b^3+27d=0$	**M1 A1**	Substitutes into third equation, AG.
		5	
1(b)	$3b=b^2$ leading to $b=3$	**M1 A1**	Uses $2\alpha^2+\gamma^2=(2\alpha+\gamma)^2-2(\alpha^2+2\alpha\gamma)$ or substitutes for α,γ in terms of b.
	$d=-4$	**A1**	
		3	

Question	Answer	Marks	Guidance
2	$7^2 + 97 - 50 = 96$ is divisible by 48	B1	Checks base case.
	Assume that $7^{2k} + 97^k - 50$ is divisible by 48 for some positive integer k.	B1	States inductive hypothesis.
	Then $7^{2k+2} + 97^{k+1} - 50 = (48+1)7^{2k} + (96+1)97^k - 50$	M1 A1	Separates $7^{2k} + 97^k - 50$ or considers difference.
	is divisible by 48 because $48 \times 7^{2k} + 96 \times 97^k$ is divisible by 48.	A1	
	Hence, by induction, true for every positive integer n.	A1	
		6	

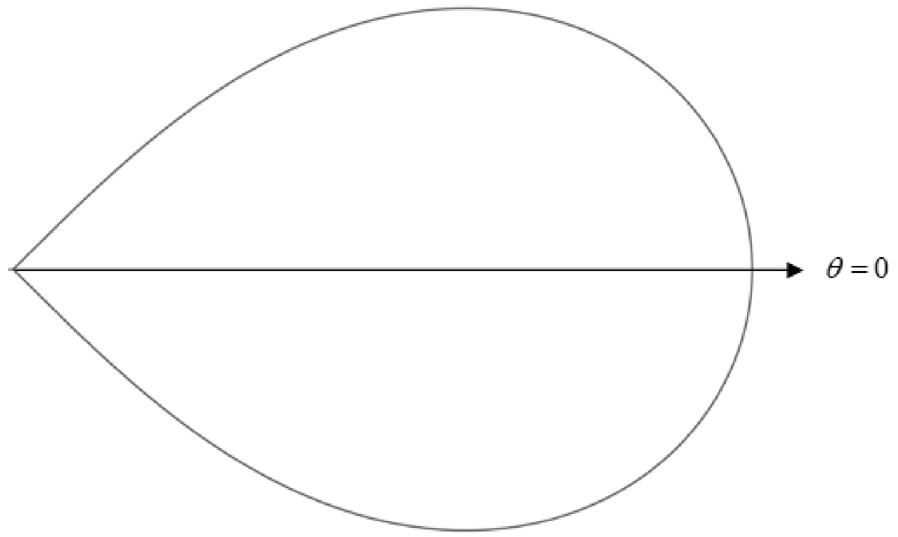
Question	Answer	Marks	Guidance
3(a)	$8r^3 + 12r^2 + 6r + 1 - \left(8r^3 - 12r^2 + 6r - 1\right) = 24r^2 + 2$	M1	Expands.
	$24\sum_{r=1}^{n} r^2 + 2n = (2n+1)^3 - 1$	M1 A1 A1	Sums both sides and uses method of differences with sufficient complete terms.
	$24\sum_{r=1}^{n} r^2 = 8n^3 + 12n^2 + 4n = 4n(n+1)(2n+1)$	A1	AG.
		5	

Question	Answer	Marks	Guidance
3(b)	$S_{2n}=\sum_{r=1}^{2n}r^2+2\sum_{r=1}^{n}(2r)^2=\sum_{r=1}^{2n}r^2+8\sum_{r=1}^{n}r^2$	M1	Relates with sum of squares.
	$S_{2n}=\frac{1}{6}(2n)(2n+1)(4n+1)+\frac{8}{6}n(n+1)(2n+1)$	M1	Applies $\sum_{r=1}^{n}r^2=\frac{1}{6}n(n+1)(2n+1)$.
	$S_{2n}=\frac{1}{3}n(2n+1)(4n+1+4n+4)=\frac{1}{3}n(2n+1)(8n+5)$	A1	$\sum_{r=1}^{n}r=\frac{1}{2}n(n+1)$.
	Alternative for 3(b)		
	$S_{2n}=\sum_{r=1}^{n}(2r-1)^2+3\sum_{r=1}^{n}(2r)^2=\sum_{r=1}^{n}(4r^2-4r+1+12r^2)$	M1	Relates with sum of squares.
	$\frac{16n(n+1)(2n+1)}{6}-4\frac{n(n+1)}{2}+n$	M1	Applies $\sum_{r=1}^{n}r^2=\frac{1}{6}n(n+1)(2n+1)$ and $\sum_{r=1}^{n}r=\frac{1}{2}n(n+1)$
	$\frac{n}{3}(16n^2+18n+5)=\frac{n}{3}(2n+1)(8n+5)$	A1	
		3	
3(c)	$\frac{16}{3}$	B1 FT	
		1	

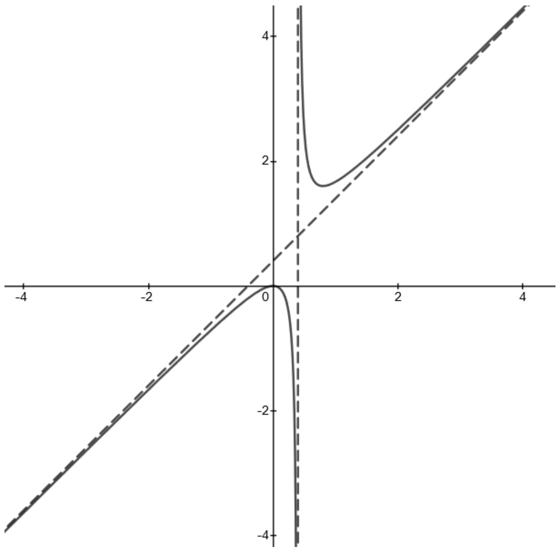
Question	Answer	Marks	Guidance
4(a)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 1 \\ 3 & 2 & -1 \end{vmatrix} = \begin{pmatrix} -4 \\ 2 \\ -8 \end{pmatrix} \sim \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$	M1 A1	Finds vector perpendicular to Π.
	$2(3)-(-2)+4(1)=12$	M1	Substitutes point on Π.
	$2x-y+4z=12$	A1	
		4	
4(b)	$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \sqrt{2}\sqrt{21}\cos\alpha$ leading to $\cos\alpha = \frac{3}{\sqrt{42}}$	M1 A1 FT	Uses dot product of $\mathbf{j}+\mathbf{k}$ and their normal. May be implied by angle 62.4°
	Acute angle between l and Π is $90-\alpha = 27.6^{\circ}$	A1	0.48 radians
		3	
4(c)	$\overrightarrow{OF} = \overrightarrow{OP} + t\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2+2t \\ 3-t \\ 1+4t \end{pmatrix}$	M1 A1	Forms $\overrightarrow{OF}$ using parameter.
	$21t+5=12$	M1	Substitutes into equation of plane.
	$\overrightarrow{OF} = \frac{1}{3}\begin{pmatrix} 8 \\ 8 \\ 7 \end{pmatrix}$	A1	
		4	

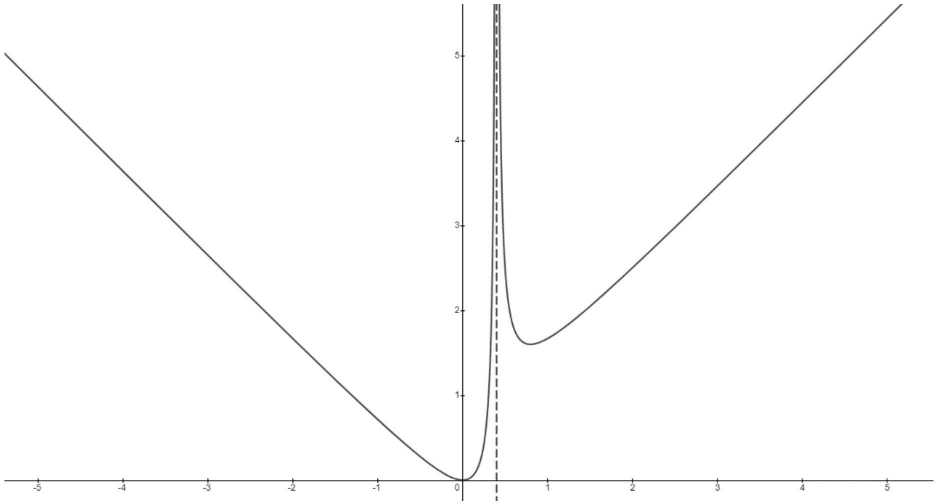
Question	Answer	Marks	Guidance
5(a)	Shear (in the x-direction) followed by a rotation (anticlockwise about the origin through 45°).	B2	Award B1 if given in the wrong order.
		2	
5(b)	$\mathbf{M}^{-1} = \frac{1}{1}\begin{pmatrix} \frac{k+1}{\sqrt{2}} & -\frac{k-1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{k+1}{\sqrt{2}} & -\frac{k-1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$	M1 A1	Finds $\mathbf{M}^{-1}$.
		2	
5(c)	$\mathbf{M} = \begin{pmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}(k-1)\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}(k+1)\sqrt{2} \end{pmatrix}$	B1	
	$\frac{1}{2}\sqrt{2}\begin{pmatrix} 1 & k-1 \\ 1 & k+1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\sqrt{2}\begin{pmatrix} x+(k-1)y \\ x+(k+1)y \end{pmatrix}$	B1	Transforms $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} X \\ Y \end{pmatrix}$.
	$x+m(k+1)x = m(x+m(k-1)x)$	M1 A1	Uses $y = mx$ and $Y = mX$.
	$(k-1)m^2 - km - 1 = 0$	A1	
	$k^2 + 4k - 4 < 0$	M1	Sets discriminant negative.
	$2(-1-\sqrt{2}) < k < 2(-1+\sqrt{2})$	A1	

Question	Answer	Marks	Guidance
5(c)	**Alternative method for question 5(c)**		
	$\mathbf{M} = \begin{pmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}(k-1)\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}(k+1)\sqrt{2} \end{pmatrix}$	**B1**	
	$[\mathbf{M} - \lambda \mathbf{I} =] \begin{vmatrix} \frac{1}{2}\sqrt{2}-\lambda & \frac{1}{2}(k-1)\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}(k+1)\sqrt{2}-\lambda \end{vmatrix} = 0$	**B1**	Sets up to find characteristic equation.
	$\left(\frac{1}{2}\sqrt{2}-\lambda\right)\left(\frac{1}{2}(k+1)\sqrt{2}-\lambda\right)-\frac{1}{2}(k-1)\sqrt{2}\,\frac{1}{2}\sqrt{2}=0$	**M1 A1**	Evaluates determinant.
	$2\lambda^2-\sqrt{2}(k+2)\lambda+2=0$	**A1**	
	$k^2+4k-4<0$	**M1**	Sets discriminant negative.
	$2\left(-1-\sqrt{2}\right)<k<2\left(-1+\sqrt{2}\right)$	**A1**	
		7	

Question	Answer	Marks	Guidance
6(a)	$x^2+y^2=r^2 \quad x=r\cos\theta \quad y=r\sin\theta$	B1	Used.
	$r^4=36r^2\left(\cos^2\theta-\sin^2\theta\right)=36r^2\cos 2\theta$	M1	Substitutes and applies $\cos 2\theta=\cos^2\theta-\sin^2\theta$.
	$r^2=36\cos 2\theta$	A1	AG.
		3	
6(b)		B1	Closed curve. Correct position and symmetrical about initial line.
		B1	Single correct loop.
	6	B1	States maximum distance or labels sketch.
		3	

Question	Answer	Marks	Guidance
6(c)	$18\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2\theta \, d\theta = 9[\sin 2\theta]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$	**M1**	Forms $\frac{1}{2}\int r^2 \, \mathrm{d}\theta$
	18	**A1**	
		2	
6(d)	$y = 6\cos^{\frac{1}{2}} 2\theta \sin\theta$	**B1**	
	$\cos^{\frac{1}{2}} 2\theta \cos\theta - \cos^{-\frac{1}{2}} 2\theta \sin 2\theta \sin\theta = 0$	**M1 A1**	Sets $\frac{\mathrm{d}y}{\mathrm{d}\theta} = 0$.
	$\cos 2\theta \cos\theta - \sin 2\theta \sin\theta = 0$ leading to $1 = \frac{2\tan^2\theta}{1-\tan^2\theta}$	**M1**	Applies suitable trigonometric identity.
	$\theta = \pm\frac{1}{6}\pi$	**A1**	
	$\frac{3}{2}\sqrt{2}$	**A1**	
		6	

Question	Answer	Marks	Guidance
7(a)	$x = \frac{2}{5}$	**B1**	Vertical asymptote.
	$y = \frac{(5x-2)(x+\frac{2}{5})+\frac{4}{5}}{5x-2}$ leading to $y = x + \frac{2}{5}$	**M1 A1**	Oblique asymptote.
		3	
7(b)	$\frac{dy}{dx} = \frac{(5x-2)(10x)-(5x^2)(5)}{(5x-2)^2}$	**M1**	Finds $\frac{dy}{dx}$.
	$5x^2 - 4x = 0$	**M1**	Sets equal to 0 and forms quadratic equation.
	$(0,0)$, $\left(\frac{4}{5}, \frac{8}{5}\right)$	**A1 A1**	
		4	
7(c)		**B1**	Axes and asymptotes.
		B1	Correct upper branch and asymptotic behaviour.
		B1	Correct lower branch.
		3	

Question	Answer	Marks	Guidance
7(d)		B1 FT	FT from sketch in part **(c).**
		B1	Correct shape as x tends to infinity.
	$\frac{5x^2}{5x-2}=2$ or $\frac{5x^2}{5x-2}=-2$ $5x^2-10x+4=0$ or $5x^2+10x-4=0$	M2	Finds critical points, award M1 for each case.
	$x=-1-\frac{3}{5}\sqrt{5},\quad x=-1+\frac{3}{5}\sqrt{5}$ or $x=1-\frac{1}{5}\sqrt{5},\quad x=1+\frac{1}{5}\sqrt{5}$	A1	Must be exact.
	$-1-\frac{3}{5}\sqrt{5}<x<-1+\frac{3}{5}\sqrt{5},\quad 1-\frac{1}{5}\sqrt{5}<x<1+\frac{1}{5}\sqrt{5}$	A1 FT	Follow through on use of decimals.
		6	

Cambridge Assessment International Education

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CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

6146662382

FURTHER MATHEMATICS **9231/12**

Paper 1 Further Pure Mathematics 1 **October/November 2022**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
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- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **20** pages. Any blank pages are indicated.

DC (LK) 303986/2

[Turn over

1 **(a)** Use the list of formulae (MF19) to find $\sum_{r=1}^{n} r(r+2)$ in terms of n, simplifying your answer. [2]

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(b) Express $\frac{1}{r(r+2)}$ in partial fractions and hence find $\sum_{r=1}^{n} \frac{1}{r(r+2)}$ in terms of n. [4]

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(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{r(r+2)}$. [1]

2 The equation $x^4 + 3x^2 + 2x + 6 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}, \frac{1}{\delta^2}$ and state the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}$. [4]

(b) Find the value of $\beta^2\gamma^2\delta^2 + \alpha^2\gamma^2\delta^2 + \alpha^2\beta^2\delta^2 + \alpha^2\beta^2\gamma^2$. [3]

(c) Find the value of $\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} + \frac{1}{\delta^4}$. [2]

3 The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$, where k is a constant and $k \neq 0$ or 1.

(a) The matrix **M** represents a sequence of two geometrical transformations.

State the type of each transformation, and make clear the order in which they are applied. [2]

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(b) Write $\mathbf{M}^{-1}$ as the product of two matrices, neither of which is **I**. [2]

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(c) Show that the invariant points of the transformation represented by **M** lie on the line $y = \frac{k^2}{1-k}x$. [4]

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(d) The triangle *ABC* in the *x*-*y* plane is transformed by **M** onto triangle *DEF*.

Find the value of k for which the area of triangle *DEF* is equal to the area of triangle *ABC*. [2]

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4 The function f is such that $f''(x) = f(x)$.

Prove by mathematical induction that, for every positive integer n,

$$\frac{d^{2n-1}}{dx^{2n-1}}\big(x\,f(x)\big) = x\,f'(x) + (2n-1)\,f(x).$$ [7]

5 The curve C has polar equation $r = a\sec^2\theta$, where a is a positive constant and $0 \leqslant \theta \leqslant \frac{1}{4}\pi$.

(a) Sketch C, stating the polar coordinates of the point of intersection of C with the initial line and also with the half-line $\theta = \frac{1}{4}\pi$. [3]

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(b) Find the maximum distance of a point of C from the initial line. [2]

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(c) Find the area of the region enclosed by C, the initial line and the half-line $\theta = \frac{1}{4}\pi$. [4]

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(d) Find, in the form $y = \mathrm{f}(x)$, the Cartesian equation of C. [3]

6 The lines l_1 and l_2 have equations $\mathbf{r} = 2\mathbf{i} + \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = 2\mathbf{j} + 6\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ respectively.

The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2.

(a) Find the length PQ. [5]

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The plane Π_1 contains PQ and l_1.

The plane Π_2 contains PQ and l_2.

(b) **(i)** Write down an equation of Π_1, giving your answer in the form $\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$. [1]

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(ii) Find an equation of Π_2, giving your answer in the form $ax + by + cz = d$. [4]

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(c) Find the acute angle between Π_1 and Π_2. [5]

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7 The curve C has equation $y = \dfrac{x^2 - x}{x + 1}$.

(a) Find the equations of the asymptotes of C. [3]

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(b) Find the exact coordinates of the stationary points on C. [4]

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(c) Sketch C, stating the coordinates of any intersections with the axes. [3]

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(d) Sketch the curve with equation $y = \left|\frac{x^2 - x}{x+1}\right|$ and find in exact form the set of values of x for which $\left|\frac{x^2 - x}{x+1}\right| < 6$. [5]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

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Paper 1 Further Pure Mathematics 1 **October/November 2022**

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Published

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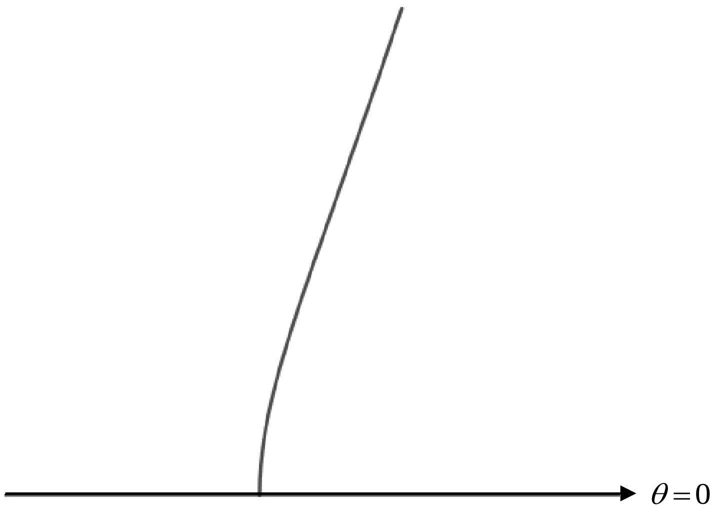
This document consists of **15** printed pages.

Question	Answer	Marks	Guidance
1(a)	$\sum_{r=1}^{n} r(r+2) = \sum_{r=1}^{n} r^2 + 2r = \frac{1}{6}n(n+1)(2n+1) + n(n+1)$	M1	Uses List of formulae (MF19).
	$\frac{1}{6}n(n+1)(2n+7)$	A1	At least all like terms collected and fractions simplified.
		2	
1(b)	$\frac{1}{r(r+2)} = \frac{1}{2}\left(\frac{1}{r} - \frac{1}{r+2}\right)$	M1 A1	Expresses in partial fractions.
	$\sum_{r=1}^{n} \frac{2}{r(r+2)} = 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} \vdots + \frac{1}{n-1} - \frac{1}{n+1} + \frac{1}{n} - \frac{1}{n+2}$	M1	Writes enough terms.
	$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{1}{2}\left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)$	A1	OE
		4	
1(c)	$\frac{3}{4}$	B1 FT	FT *their* part **(b)** of correct form.
		1	

Question	Answer	Marks	Guidance
2(a)	$y = x^{-2}$ leading to $x = y^{-\frac{1}{2}}$	B1	Correct substitution.
	$y^{-2} + 3y^{-1} + 2y^{-\frac{1}{2}} + 6 = 0$ leading to $1 + 3y + 2y^{\frac{3}{2}} + 6y^2 = 0$	M1	Obtains an equation not involving radicals.
	$36y^4 + 32y^3 + 21y^2 + 6y + 1 = 0$	A1	
	$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2} = -\frac{32}{36} = -\frac{8}{9}$	B1 FT	
		4	
2(b)	$\alpha\beta\gamma\delta = 6$	B1	SOI
	$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2} = \frac{\alpha^2\beta^2\delta^2 + \alpha^2\beta^2\gamma^2 + \beta^2\gamma^2\delta^2 + \alpha^2\gamma^2\delta^2}{\alpha^2\beta^2\gamma^2\delta^2}$	M1	Relates to coefficients.
	$\beta^2\gamma^2\delta^2 + \alpha^2\gamma^2\delta^2\alpha^2\beta^2\delta^2 + \alpha^2\beta^2\gamma^2 = -32$	A1	
		3	
2(c)	$\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} + \frac{1}{\delta^4} = \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}\right)^2$ $-2\left(\alpha^{-2}\beta^{-2} + \alpha^{-2}\gamma^{-2} + a^{-2}\delta^{-2} + \beta^{-2}\gamma^{-2} + \beta^{-2}\delta^{-2} + \gamma^{-2}\delta^{-2}\right)$ $= \left(-\frac{8}{9}\right)^2 - 2\left(\frac{21}{36}\right)$	M1	Uses formula for sum of squares.
	$-\frac{61}{162}$	A1	
		2	

Question	Answer	Marks	Guidance
3(a)	Shear [in the *y*-direction] followed by a stretch, [parallel to the *y*-axis, scale factor *k*].	**B2**	Award B1 if given in the wrong order.
		2	
3(b)	$\begin{pmatrix}1&0\\k&1\end{pmatrix}^{-1}=\begin{pmatrix}1&0\\-k&1\end{pmatrix}$, $\begin{pmatrix}1&0\\0&k\end{pmatrix}^{-1}=\begin{pmatrix}1&0\\0&\frac{1}{k}\end{pmatrix}$	**B1**	
	$\mathbf{M}^{-1}=\begin{pmatrix}1&0\\-k&1\end{pmatrix}\begin{pmatrix}1&0\\0&\frac{1}{k}\end{pmatrix}$	**B1**	Correct order.
		2	
3(c)	$\mathbf{M}=\begin{pmatrix}1&0\\k^2&k\end{pmatrix}$	**B1**	
	$\begin{pmatrix}1&0\\k^2&k\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\k^2x+ky\end{pmatrix}$	**B1**	Transforms $\begin{pmatrix}x\\y\end{pmatrix}$ to $\begin{pmatrix}X\\Y\end{pmatrix}$
	$k^2x+ky=y$	**M1**	Sets $\begin{pmatrix}X\\Y\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$
	$y=\dfrac{k^2}{1-k}x$	**A1**	AG.
		4	
3(d)	$1=\lvert k\rvert$	**M1**	Uses $\lvert DEF\rvert=\lvert\det\mathbf{M}\rvert\lvert ABC\rvert$
	$k=-1$	**A1**	
		2	

Question	Answer	Marks	Guidance
4	$\frac{\mathrm{d}}{\mathrm{d}x}\left(xf(x)\right) = xf'(x) + f(x) = xf'(x) + (2(1)-1)f(x)$	**B1**	Checks base case using product rule.
	Assume true for $n = k$, so $\frac{\mathrm{d}^{2k-1}}{\mathrm{d}x^{2k-1}}\left(xf(x)\right) = xf'(x) + (2k-1)f(x)$	**B1**	States inductive hypothesis.
	Then $\frac{\mathrm{d}^{2k}}{\mathrm{d}x^{2k}}\left(xf(x)\right) = xf(x) + 2kf'(x)$	**M1 A1**	Differentiates once.
	$\frac{\mathrm{d}^{2k+1}}{\mathrm{d}x^{2k+1}}\left(xf(x)\right) = xf'(x) + f(x) + 2kf(x)$ $= xf'(x) + (2k+1)f(x)$	**M1 A1**	Differentiates again.
	So, it is also true for $n = k+1$. Hence, by induction, true for all positive integers.	**A1**	States conclusion.
		7	

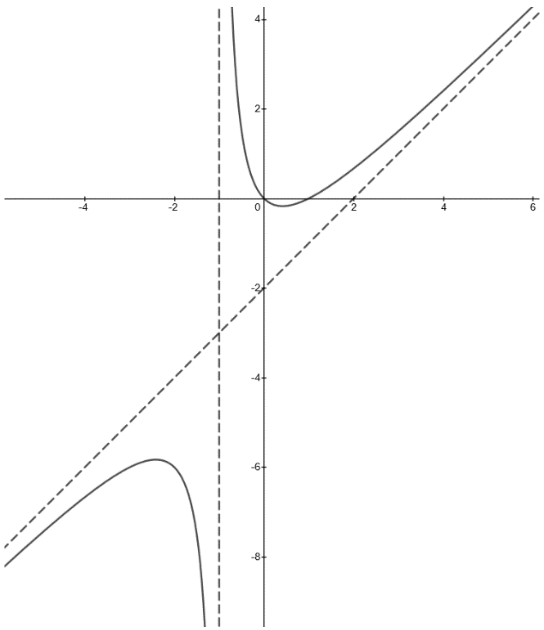
Question	Answer	Marks	Guidance
5(a)		B1	Correct position and domain and r strictly increasing.
		B1	Decreasing positive gradient.
	$(a, 0)$ $(2a, \frac{1}{4}\pi)$	B1	Can be labelled on their sketch.
		3	
5(b)	$y = (2a)\sin\left(\frac{1}{4}\pi\right)$	M1	Uses $y = r\sin\theta$
	$a\sqrt{2}$	A1	
		2	

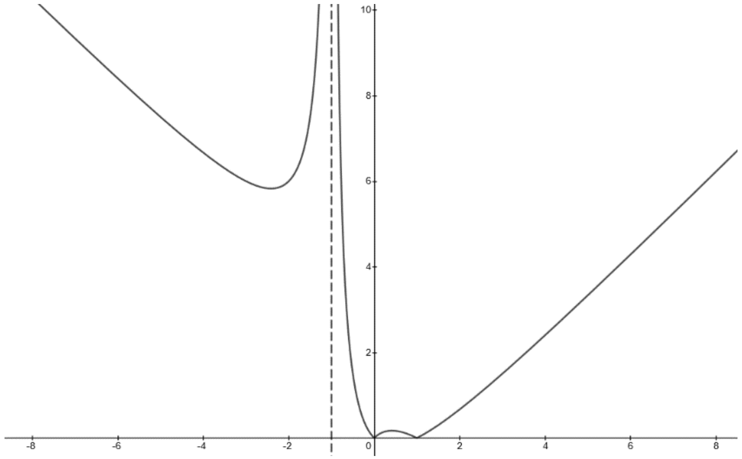
Question	Answer	Marks	Guidance
5(c)	$\frac{1}{2}a^2\int_0^{\frac{1}{4}\pi}\sec^4\theta\,\mathrm{d}\theta$	M1	Forms $\frac{1}{2}\int r^2\,\mathrm{d}\theta$ with correct limits.
	$=\frac{1}{2}a^2\int_0^{\frac{1}{4}\pi}\sec^2\theta\left(\tan^2\theta+1\right)\mathrm{d}\theta$	M1	Applies $\sec^2\theta=\tan^2\theta+1$ to obtain an integrable form.
	$=\frac{1}{2}a^2\left[\frac{1}{3}\tan^3\theta+\tan\theta\right]_0^{\frac{1}{4}\pi}$	A1	
	$=\frac{2}{3}a^2$	A1	
		4	
5(d)	$x^2+y^2=r^2 \quad x=r\cos\theta$	M1	Substitutes to eliminate either r or θ.
	$r^2\cos^2\theta=ar$ leading to $x^2=a\sqrt{x^2+y^2}$ leading to $x^4=a^2\left(x^2+y^2\right)$	M1	Completes substitution to eliminate both r and θ and makes y or y^2 the subject.
	$y=\sqrt{a^{-2}x^4-x^2}=x\sqrt{a^{-2}x^2-1}$	A1	AEF
		3	

Question	Answer	Marks	Guidance
6(a)	$\begin{pmatrix}0\\2\\6\end{pmatrix}-\begin{pmatrix}2\\0\\1\end{pmatrix}=\begin{pmatrix}-2\\2\\5\end{pmatrix}$	**B1**	Finds a vector joining any point of l_1 to any point of l_2.
.	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&-2\\1&-1&2\end{vmatrix}=\begin{pmatrix}2\\-4\\-3\end{pmatrix}$	**M1 A1**	Finds common perpendicular.
	$\frac{1}{\sqrt{29}}\left\|\begin{pmatrix}-2\\2\\5\end{pmatrix}\bullet\begin{pmatrix}2\\-4\\-3\end{pmatrix}\right\|=\frac{27}{\sqrt{29}}=5.01$	**M1 A1**	Uses formula for shortest distance.
		5	
6(b)(i)	$\mathbf{r}=2\mathbf{i}+\mathbf{k}+s(\mathbf{i}-\mathbf{j}+2\mathbf{k})+t(2\mathbf{i}-4\mathbf{j}-3\mathbf{k})$	**B1**	FT *their* common perpendicular from part **(a)**.
		1	
6(b)(ii)	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&-2\\2&-4&-3\end{vmatrix}=\begin{pmatrix}-14\\-1\\-8\end{pmatrix}$	**M1 A1FT**	Finds normal to the plane Π_2. FT *their* common perpendicular from part **(a)**.
	$14(0)+(2)+8(6)=50\Rightarrow 14x+y+8z=50$	**M1 A1**	Substitutes point.
		4	

Question	Answer	Marks	Guidance
6(c)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 2 & -4 & -3 \end{vmatrix} = \begin{pmatrix} 11 \\ 7 \\ -2 \end{pmatrix}$	**M1 A1 FT**	Finds normal to the plane Π_1. FT *their* part **(b)(i).**
	$\begin{pmatrix} 11 \\ 7 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 14 \\ 1 \\ 8 \end{pmatrix} = \sqrt{174}\sqrt{261}\cos\theta$ leading to $\cos\theta = \frac{145}{\sqrt{174}\sqrt{261}}$	**M1 A1**	Uses dot product of normal vectors.
	47.1°	**A1**	0.822 radians
		5	

Question	Answer	Marks	Guidance
7(a)	$x = -1$	**B1**	States vertical asymptote.
	$y = \frac{(x+1)(x-2)+2}{x+1}$	**M1**	Finds oblique asymptote.
	$y = x - 2$	**A1**	
		3	
7(b)	$\frac{dy}{dx} = 1 - 2(x+1)^{-2} = 0 \Rightarrow (x+1)^2 = 2$	**M1 A1**	Differentiates and sets derivative equal to 0.
	$\left(-1+\sqrt{2}, -3+2\sqrt{2}\right)$, $\left(-1-\sqrt{2}, -3-2\sqrt{2}\right)$	**A1 A1**	
		4	

Question	Answer	Marks	Guidance
7(c)		B1	Axes and asymptotes labelled.
		B1	Upper branch with $(0,0)$ and $(1,0)$ stated or clear on scale.
		B1	Lower branch correct and all asymptotic approaches correct.
		3	

Question	Answer	Marks	Guidance
7(d)		B1 FT	FT from sketch in part (c).
	$\dfrac{x^2-x}{x+1}=6$ or $\dfrac{x^2-x}{x+1}=-6$ $x^2-7x-6=0$ or $x^2+5x+6=0$	M2	Finds critical points, award M1 for each case.
	$x=-3,-2,\ \frac{7}{2}-\frac{1}{2}\sqrt{73},\ \frac{7}{2}+\frac{1}{2}\sqrt{73}$	A1	
	$-3<x<-2$ and $\frac{7}{2}-\frac{1}{2}\sqrt{73}<x<\frac{7}{2}+\frac{1}{2}\sqrt{73}$	A1	
		5	

Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

4190141506

FURTHER MATHEMATICS **9231/11**

Paper 1 Further Pure Mathematics 1 **May/June 2022**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **20** pages. Any blank pages are indicated.

DC (JP) 311983

1 Let a be a positive constant.

(a) Use the method of differences to find $\displaystyle\sum_{r=1}^{n} \frac{1}{(ar+1)(ar+a+1)}$ in terms of n and a. [4]

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(b) Find the value of a for which $\displaystyle\sum_{r=1}^{\infty} \frac{1}{(ar+1)(ar+a+1)} = \frac{1}{6}$. [3]

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2 The points A, B, C have position vectors

$$4\mathbf{i} - 4\mathbf{j} + \mathbf{k}, \quad -4\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}, \quad 4\mathbf{i} - \mathbf{j} - 2\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax + by + cz = d$. [5]

(b) Find the perpendicular distance from O to the plane ABC. [2]

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(c) The point D has position vector $2\mathbf{i}+3\mathbf{j}-3\mathbf{k}$.

Find the coordinates of the point of intersection of the line OD with the plane ABC. [3]

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3 The sequence of positive numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 > 4$ and, for $n \geqslant 1$,

$$u_{n+1} = \frac{u_n^2 + u_n + 12}{2u_n}.$$

(a) By considering $u_{n+1} - 4$, or otherwise, prove by mathematical induction that $u_n > 4$ for all positive integers n. [5]

(b) Show that $u_{n+1} < u_n$ for $n \geqslant 1$. [3]

4 The cubic equation $2x^3 + 5x^2 - 6 = 0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\frac{1}{\alpha^3}, \frac{1}{\beta^3}, \frac{1}{\gamma^3}$. [3]

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(b) Find the value of $\frac{1}{\alpha^6} + \frac{1}{\beta^6} + \frac{1}{\gamma^6}$. [3]

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(c) Find also the value of $\frac{1}{\alpha^9} + \frac{1}{\beta^9} + \frac{1}{\gamma^9}$. [2]

5 The curve C has equation $y = \dfrac{2x^2 - x - 1}{x^2 + x + 1}$.

(a) Show that C has no vertical asymptotes and state the equation of the horizontal asymptote of C. [3]

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(b) Find the coordinates of the stationary points on C. [4]

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(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

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(d) Sketch the curve with equation $y = \left|\dfrac{2x^2 - x - 1}{x^2 + x + 1}\right|$ and state the set of values of k for which $\left|\dfrac{2x^2 - x - 1}{x^2 + x + 1}\right| = k$ has 4 distinct real solutions. [2]

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6 The curve C has polar equation $r^2 = \tan^{-1}\left(\frac{1}{2}\theta\right)$, where $0 \leqslant \theta \leqslant 2$.

(a) Sketch C and state, in exact form, the greatest distance of a point on C from the pole. [3]

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(b) Find the exact value of the area of the region bounded by C and the half-line $\theta = 2$. [5]

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Now consider the part of C where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(c) Show that, at the point furthest from the half-line $\theta = \frac{1}{2}\pi$,

$$(\theta^2 + 4)\tan^{-1}\left(\tfrac{1}{2}\theta\right)\sin\theta - \cos\theta = 0$$

and verify that this equation has a root between 0.6 and 0.7. [5]

7 The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & k & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

(a) Find the set of values of k for which **A** is non-singular. [3]

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(b) Given that **A** is non-singular, find, in terms of k, the entries in the top row of $\mathbf{A}^{-1}$. [4]

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(c) Given that $\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, give an example of a matrix $\mathbf{C}$ such that $\mathbf{BAC} = \begin{pmatrix} 2 & 1 \\ k & 4 \end{pmatrix}$. [4]

(d) Find the set of values of k for which the transformation in the x-y plane represented by $\begin{pmatrix} 2 & 1 \\ k & 4 \end{pmatrix}$ has two distinct invariant lines through the origin. [6]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

Cambridge Assessment International Education

Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/11**

Paper 1 Further Pure Mathematics 1 **May/June 2022**

MARK SCHEME

Maximum Mark: 75

Published

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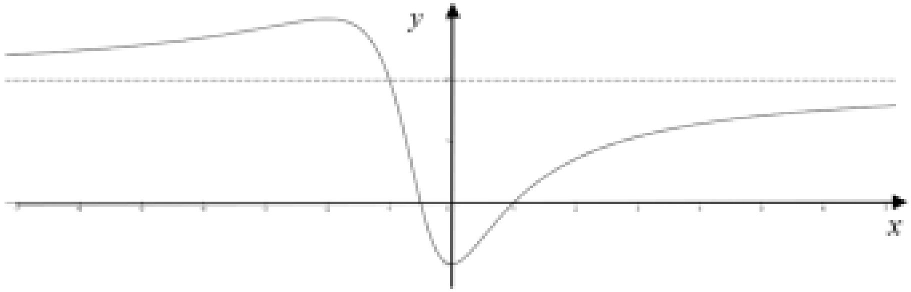
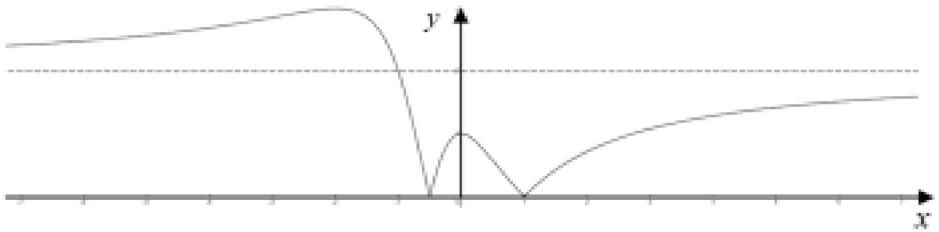
1(a)	$\dfrac{1}{(ar+1)(ar+a+1)} = \dfrac{1}{a}\left(\dfrac{1}{ar+1} - \dfrac{1}{ar+a+1}\right)$	M1 A1	Finds partial fractions.
	$\sum_{r=1}^{n} \dfrac{1}{(ar+1)(ar+a+1)} = \dfrac{1}{a}\left(\dfrac{1}{a+1} - \dfrac{1}{2a+1} + \dfrac{1}{2a+1} - \dfrac{1}{3a+1} + \ldots + \dfrac{1}{an+1} - \dfrac{1}{an+a+1}\right)$	M1	Writes at least three complete terms, including first and last.
	$\sum_{r=1}^{n} \dfrac{1}{(ar+1)(ar+a+1)} = \dfrac{1}{a}\left(\dfrac{1}{a+1} - \dfrac{1}{an+a+1}\right)$	A1	OE ISW
		4	
1(b)	$\sum_{r=1}^{\infty} \dfrac{1}{(ar+1)(ar+a+1)} = \dfrac{1}{a^2+a} = \dfrac{1}{6}$ leading to $a^2 + a - 6 = 0$	M1 A1	Finds sum to infinity, forms quadratic.
	$a = 2$	A1	CAO
		3	

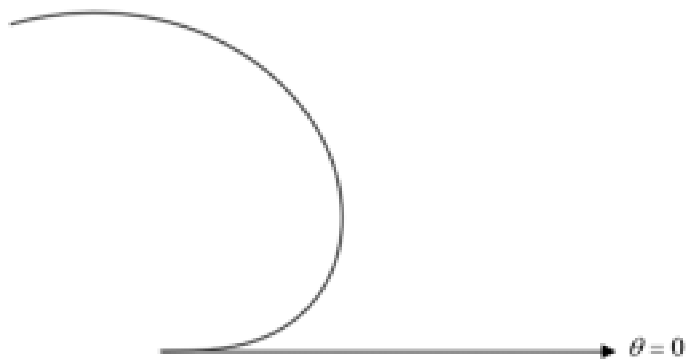
Question	Answer	Marks	Guidance
2(a)	$\overrightarrow{AB} = -8\mathbf{i} + 7\mathbf{j} - 5\mathbf{k}$ $\quad \overrightarrow{AC} = 3\mathbf{j} - 3\mathbf{k}$	B1	Finds direction vectors of two lines in the plane.
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -8 & 7 & -5 \\ 0 & 3 & -3 \end{vmatrix} = \begin{pmatrix} -6 \\ -24 \\ -24 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}$	M1 A1	Finds normal to the plane *ABC*.
	$(4) + 4(-4) + 4(1) = -8 \Rightarrow x + 4y + 4z = -8$	M1 A1	Substitutes point.
	Alternative method for question 2(a)		
	Setting up 3 equations using the points given	M1	
	$x + 4y + 4z = -8$	A1 A1 A1 A1	
		5	
2(b)	$\dfrac{8}{\sqrt{1^2 + 4^2 + 4^2}} = 1.39$	M1 A1	Divides *their* constant by magnitude of *their* normal vector. $\dfrac{8}{\sqrt{33}}$ CAO
		2	
2(c)	$\mathbf{r} = t\begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix}$	B1	Equation of line *OD*.
	$2t + 12t - 12t = -8$	M1	Substitutes into equation of plane.
	$(-8, -12, 12)$	A1	
		3	

Question	Answer	Marks	Guidance
3(a)	$u_1 > 4$ (given)	B1	States base case.
	Assume that $u_k > 4$ for some positive integer k.	B1	States inductive hypothesis.
	Then $u_{k+1} - 4 = \frac{u_k^{\,2} + u_k + 12}{2u_k} - 4 = \frac{u_k^{\,2} - 7u_k + 12}{2u_k}$	M1	Considers $u_{k+1} - 4$, puts over common denominator.
	$u_{k+1} - 4 = \frac{(u_k - 3)(u_k - 4)}{2u_k} > 0$	A1	
	Hence, by induction, $u_n > 4$ for all positive integers n.	A1	
		5	
3(b)	$u_{n+1} - u_n = \frac{-u_n^{\,2} + u_n + 12}{2u_n} = \frac{-(u_n - 4)(u_n + 3)}{2u_n}$	M1 A1	Considers $u_{n+1} - u_n$.
	So $u_n > 4 \Rightarrow u_{n+1} - u_n < 0$	A1	Uses $u_n > 4$.
		3	

Question	Answer	Marks	Guidance
4(a)	$y = x^{-3} \Rightarrow x = y^{-\frac{1}{3}}$	B1	Substitutes.
	$\Rightarrow 2y^{-1} + 5y^{-\frac{2}{3}} - 6 = 0$ leading to $5y^{-\frac{2}{3}} = 6 - 2y^{-1} \Rightarrow 125y = (6y-2)^3$	M1	Cubes to eliminate radical.
	$216y^3 - 216y^2 - 53y - 8 = 0$	A1	
	Alternative method for question 4(a)		
	$(2x^3-6)^3 = (-5x^2)^3$	M1	$8x^9 - 72x^6 + 125x^6 + 216x^3 - 216 = 0$
	$y = x^{-3}$ leading to $x^3 = y^{-1}$	B1	Substitutes.
	$216y^3 - 216y^2 - 53y - 8 = 0$	A1	
		3	
4(b)	$\alpha^{-3} + \beta^{-3} + \gamma^{-3} = 1 \qquad \alpha^{-3}\beta^{-3} + \beta^{-3}\gamma^{-3} + \gamma^{-3}\alpha^{-3} = -\frac{53}{216}$	B1 FT	Using *their* answer to part **(a)**.
	$\alpha^{-6} + \beta^{-6} + \gamma^{-6} = 1^2 - 2\left(-\frac{53}{216}\right)$	M1	$\alpha^{-6} + \beta^{-6} + \gamma^{-6} = (\alpha^{-3} + \beta^{-3} + \gamma^{-3})^2 - 2(\alpha^{-3}\beta^{-3} + \beta^{-3}\gamma^{-3} + \gamma^{-3}\alpha^{-3})$
	$\alpha^{-6} + \beta^{-6} + \gamma^{-6} = \frac{161}{108}$	A1	
		3	
4(c)	$216S_{-9} = 216S_{-6} + 53S_{-3} + 24$	M1	Using *their* $216\alpha^{-9} - 216\alpha^{-6} - 53\alpha^{-3} - 8 = 0$
	$S_{-9} = \frac{399}{216} = \frac{133}{72}$	A1	
		2	

Question	Answer	Marks	Guidance
5(a)	$-3<0$	M1 A1	Finds discriminant or roots of x^2+x+1, or completes square.
	$y=2$	B1	Horizontal asymptote.
		3	
5(b)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(x^2+x+1)(4x-1)-(2x^2-x-1)(2x+1)}{\left(x^2+x+1\right)^2}$	M1	Finds $\frac{\mathrm{d}y}{\mathrm{d}x}$.
	$3x^2+6x=0$	M1	Sets equal to 0 and forms equation.
	$(0,-1)$ $(-2,3)$	A1 A1	One point correct, or both x values. Other point correct.
	Alternative method for question 5(b)		
	$2x^2-x-1-y(x^2+x+1)=0$	M1	Forms quadratic equation
	Finds discriminant $(y+1)^2-4(y+1)(y-2)$ AND states y exists if discriminant ≥ 0 OR does not exist if discriminant < 0	M1	
	Finds (0, -1) and (-2, 3)	A1	
	Explains why they are stationary values.	A1	Double x roots for $y=-1$ and $y=3$ or no vertical asymptote etc
		4	

Question	Answer	Marks	Guidance
5(c)		B1	Axes and asymptote.
		B1	Correct shape and position.
	$(1,0)$, $\left(-\frac{1}{2},0\right)$, $(0,-1)$	B1	States coordinates of intersections with axes.
		3	
5(d)		B1 FT	FT from sketch in part **(c)**. There must be cusps on the x axis.
	$0<k<1$	B1	
		2	

Question	Answer	Marks	Guidance
6(a)		B1	Correct domain, r strictly increasing.
		B1	Correct gradient when $\theta = 0$ and $\theta = 2$.
	Maximum distance of C from the pole is $\sqrt{\frac{1}{4}\pi}$.	B1	Must be exact. $\frac{1}{2}\sqrt{\pi}$
		3	
6(b)	$\frac{1}{2}\int_0^2 \tan^{-1}\left(\frac{1}{2}\theta\right)\mathrm{d}\theta$	M1	Uses $\frac{1}{2}\int r^2\,\mathrm{d}\theta$ with correct limits.
	$\left[\frac{1}{2}\theta\tan^{-1}\left(\frac{1}{2}\theta\right)\right]_0^2 - \int_0^2 \frac{\theta}{\theta^2+4}\mathrm{d}\theta$	M1 A1	Applies integration by parts.
	$\frac{1}{2}\left[\theta\tan^{-1}\left(\frac{1}{2}\theta\right) - \ln(\theta^2+4)\right]_0^2$	A1	
	$\frac{1}{4}\pi - \frac{1}{2}\ln 8 + \frac{1}{2}\ln 4 = \frac{1}{4}\pi - \frac{1}{2}\ln 2$	A1	Must be exact.
		5	

Question	Answer	Marks	Guidance
6(c)	$x=\left(\tan^{-1}\left(\frac{1}{2}\theta\right)\right)^{\frac{1}{2}}\cos\theta$	B1	Uses $x=r\cos\theta$
	$\frac{dx}{d\theta}=-\left(\tan^{-1}\left(\frac{1}{2}\theta\right)\right)^{\frac{1}{2}}\sin\theta+\cos\theta\left(\tan^{-1}\left(\frac{1}{2}\theta\right)\right)^{-\frac{1}{2}}\left(\theta^2+4\right)^{-1}=0$	M1 A1	Sets derivative equal to zero.
	$\left(\theta^2+4\right)\tan^{-1}\left(\frac{1}{2}\theta\right)\sin\theta-\cos\theta=0$	A1	AG
	$\left(0.6^2+4\right)\tan^{-1}(0.3)\sin 0.6-\cos 0.6=-0.108$ and $\left(0.7^2+4\right)\tan^{-1}(0.35)\sin 0.7-\cos 0.7=0.209$	B1	Shows sign change.
		5	

Question	Answer	Marks	Guidance
7(a)	$\begin{vmatrix} k & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & k \\ 7 & 8 \end{vmatrix} = 0 \Rightarrow -12k + 60 = 0$	M1 A1	Sets $\det \mathbf{A} = 0$. SOI
	$k \neq 5$	A1	
		3	
7(b)	$\dfrac{1}{60-12k}\begin{vmatrix} k & 6 \\ 8 & 9 \end{vmatrix} = \dfrac{9k-48}{60-12k} = \dfrac{3k-16}{4(5-k)}$	M1 A1	Finds one correct entry.
	$\dfrac{-1}{60-12k}\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = \dfrac{6}{60-12k} = \dfrac{1}{2(5-k)}$	A1	
	$\dfrac{1}{60-12k}\begin{vmatrix} 2 & 3 \\ k & 6 \end{vmatrix} = \dfrac{12-3k}{60-12k} = \dfrac{4-k}{4(5-k)}$	A1	
		4	

Question	Answer	Marks	Guidance
7(c)	$\mathbf{BA}=\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}1&2&3\\4&k&6\\7&8&9\end{pmatrix}=\begin{pmatrix}1&2&3\\4&k&6\end{pmatrix}$	M1 A1	Finds **BA**.
	$\begin{pmatrix}1&2&3\\4&k&6\end{pmatrix}\begin{pmatrix}a&d\\b&e\\c&f\end{pmatrix}=\begin{pmatrix}a+2b+3c&d+2e+3f\\4a+kb+6c&4d+ke+6f\end{pmatrix}=\begin{pmatrix}2&1\\k&4\end{pmatrix}$	M1	Sets **C** as 3×2 matrix.
	$\mathbf{C}=\begin{pmatrix}0&1\\1&0\\0&0\end{pmatrix}$	A1	Accept any *a, b, c, d, e* and *f* such that $\begin{pmatrix}a+2b+3c&d+2e+3f\\4a+kb+6c&4d+ke+6f\end{pmatrix}=\begin{pmatrix}2&1\\k&4\end{pmatrix}$.
		4	
7(d)	$\begin{pmatrix}2&1\\k&4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2x+y\\kx+4y\end{pmatrix}$	B1	Transforms $\begin{pmatrix}x\\y\end{pmatrix}$ to $\begin{pmatrix}X\\Y\end{pmatrix}$.
	$kx+4mx=m(2x+mx)$	M1 A1	Uses $y=mx$ and $Y=mX$.
	$m^2-2m-k=0$	A1	
	$4+4k>0$	M1	Sets discriminant positive.
	$k>-1 \quad k\neq8$	A1	
		6	

Cambridge Assessment International Education

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2213454579

FURTHER MATHEMATICS **9231/13**

Paper 1 Further Pure Mathematics 1 **May/June 2022**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **20** pages. Any blank pages are indicated.

DC (CE) 214265/1

1 **(a)** Sketch the curve with equation $y = \dfrac{x+1}{x-1}$. [2]

(b) Sketch the curve with equation $y = \dfrac{|x|+1}{|x|-1}$ and find the set of values of x for which $\dfrac{|x|+1}{|x|-1} < -2$. [4]

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2 The cubic equation $x^3 + 5x^2 + 10x - 2 = 0$ has roots α, β, γ.

(a) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [3]

(b) Show that the matrix $\begin{pmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{pmatrix}$ is singular. [4]

3 A curve C has equation $y = \dfrac{ax^2 + x - 1}{x - 1}$, where a is a positive constant.

(a) Find the equations of the asymptotes of C. [3]

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(b) Show that there is no point on C for which $1 < y < 1 + 4a$. [4]

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(c) Sketch *C*. You do not need to find the coordinates of the intersections with the axes. [3]

4 Let $u_r = e^{rx}(e^{2x} - 2e^x + 1)$.

(a) Using the method of differences, or otherwise, find $\sum_{r=1}^{n} u_r$ in terms of n and x. [3]

(b) Deduce the set of non-zero values of x for which the infinite series

$$u_1 + u_2 + u_3 + \dots$$

is convergent and give the sum to infinity when this exists. [3]

(c) Using a standard result from the list of formulae (MF19), find $\sum_{r=1}^{n} \ln u_r$ in terms of n and x. [3]

5 Let $\mathbf{A} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, where a is a positive constant.

(a) State the type of the geometrical transformation in the x–y plane represented by **A**. [1]

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(b) Prove by mathematical induction that, for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}.$$ [5]

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Let $\mathbf{B} = \begin{pmatrix} b & b \\ a^{-1} & a^{-1} \end{pmatrix}$, where b is a positive constant.

(c) Find the equations of the invariant lines, through the origin, of the transformation in the x–y plane represented by $\mathbf{A}^n\mathbf{B}$. [6]

6 The curve C has Cartesian equation $x^2 + xy + y^2 = a$, where a is a positive constant.

(a) Show that the polar equation of C is $r^2 = \dfrac{2a}{2 + \sin 2\theta}$. [3]

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(b) Sketch the part of C for $0 \leqslant \theta \leqslant \frac{1}{4}\pi$. [2]

The region R is enclosed by this part of C, the initial line and the half-line $\theta = \frac{1}{4}\pi$.

(c) It is given that $\sin 2\theta$ may be expressed as $\dfrac{2\tan\theta}{1+\tan^2\theta}$. Use this result to show that the area of R is

$$\frac{1}{2}a\int_0^{\frac{1}{4}\pi} \frac{1+\tan^2\theta}{1+\tan\theta+\tan^2\theta}\,d\theta$$

and use the substitution $t = \tan\theta$ to find the exact value of this area. [8]

7 The position vectors of the points A, B, C, D are

$$7\mathbf{i}+4\mathbf{j}-\mathbf{k}, \qquad 11\mathbf{i}+3\mathbf{j}, \qquad 2\mathbf{i}+6\mathbf{j}+3\mathbf{k}, \qquad 2\mathbf{i}+7\mathbf{j}+\lambda\mathbf{k}$$

respectively.

(a) Given that the shortest distance between the line AB and the line CD is 3, show that $\lambda^2-5\lambda+4=0$. [7]

Let Π_1 be the plane ABD when $\lambda = 1$.

Let Π_2 be the plane ABD when $\lambda = 4$.

(b) **(i)** Write down an equation of Π_1, giving your answer in the form $\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$. [2]

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(ii) Find an equation of Π_2, giving your answer in the form $ax + by + cz = d$. [4]

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(c) Find the acute angle between Π_1 and Π_2. [5]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

Cambridge Assessment International Education

Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/13**

Paper 1 Further Pure Mathematics 1 **May/June 2022**

MARK SCHEME

Maximum Mark: 75

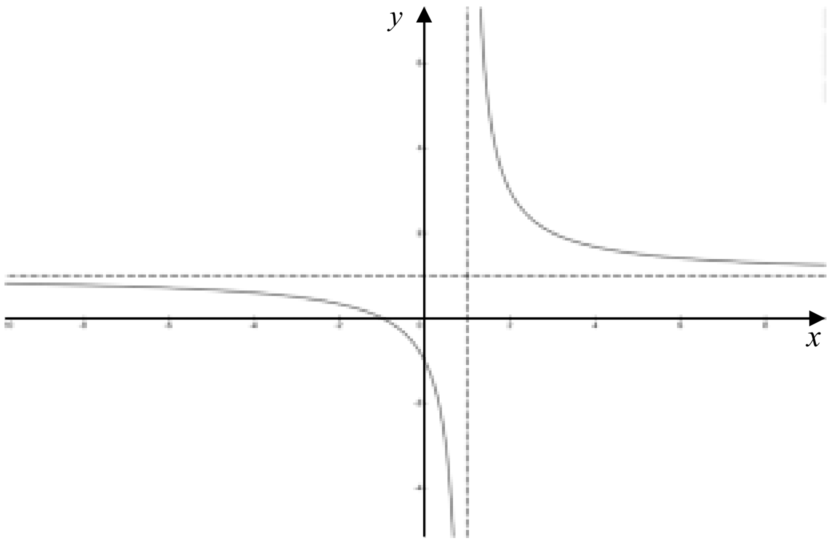
Published

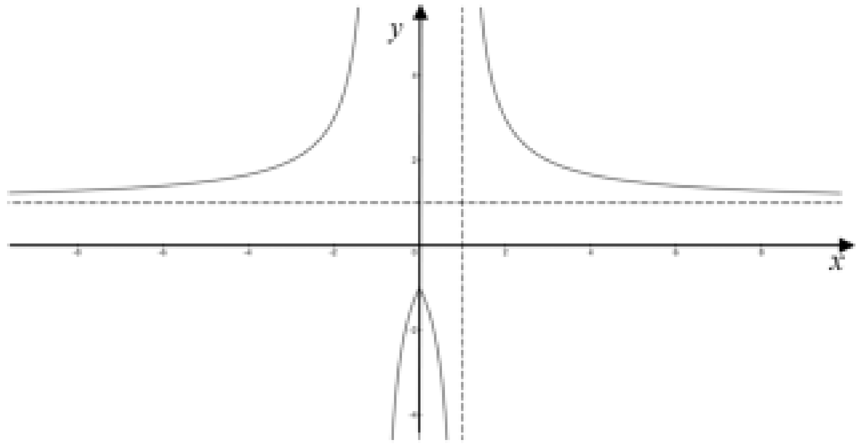
This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the May/June 2022 series for most Cambridge IGCSE, Cambridge International A and AS Level and Cambridge Pre-U components, and some Cambridge O Level components.

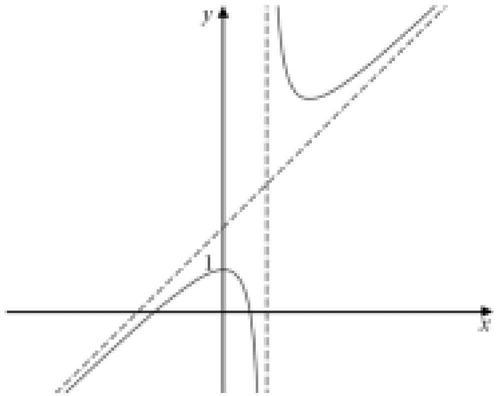
Question	Answer	Marks	Guidance
1(a)		B1	Axes and asymptotes.
		B1	Branches correct.
		2	

Question	Answer	Marks	Guidance
1(b)		B1 FT	FT from sketch in part **(a)**, symmetrical about $x = 0$.
		B1	Correct shape at $x = 0$ (reflection not turning point).
	$\frac{\lvert x\rvert + 1}{\lvert x\rvert - 1} = -2$ leading to $\lvert x\rvert = \frac{1}{3}$	M1	Finds critical point(s).
	$-1 < x < -\frac{1}{3},\ \frac{1}{3} < x < 1$	A1	
		4	

Question	Answer	Marks	Guidance
2(a)	$\alpha\beta+\beta\gamma+\gamma\alpha=10$	B1	SOI
	$\alpha^2+\beta^2+\gamma^2=5^2-20=5$	M1 A1	Uses $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$.
	Alternative method for question 2(a)		
	Put $y=x^2$ and form cubic not involving x	M1 A1	$y(y+10)^2=25y^2-20y+4$ (need not be simplified). $y^3-5y^2+120y-4=0$
	$\alpha^2+\beta^2+\gamma^2=[-(-5)]=5$	A1	
		3	
2(b)	$\begin{vmatrix}1&\alpha&\beta\\\alpha&1&\gamma\\\beta&\gamma&1\end{vmatrix}=\begin{vmatrix}1&\gamma\\\gamma&1\end{vmatrix}-\alpha\begin{vmatrix}\alpha&\gamma\\\beta&1\end{vmatrix}+\beta\begin{vmatrix}\alpha&1\\\beta&\gamma\end{vmatrix}=1-\alpha^2-\beta^2-\gamma^2+2\alpha\beta\gamma$	M1 A1	Finds determinant.
	$1-\alpha^2-\beta^2-\gamma^2+2\alpha\beta\gamma=1-5+2(2)=0$	M1 A1	Substitutes and uses $\alpha\beta\gamma=2$
		4	

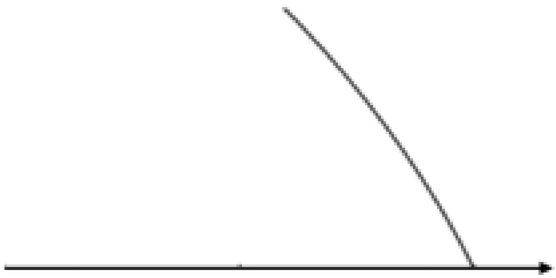
Question	Answer	Marks	Guidance
3(a)	$x=1$	**B1**	States vertical asymptote.
	$y=\dfrac{(x-1)(ax+a+1)+a}{x-1}=ax+a+1+\dfrac{a}{x-1}$	**M1**	Finds oblique asymptote.
	$y=ax+a+1$	**A1**	
		3	

Question	Answer	Marks	Guidance
3(b)	$ax^2 + x - 1 = yx - y$ leading to $ax^2 - (y-1)x + (y-1) = 0$	M1 A1	Forms a three term quadratic in x.
	$(y-1)^2 - 4a(y-1) < 0$	M1	Correct inequality using discriminant.
	$1 < y < 1 + 4a$	A1	AG Clear method to reach given answer eg $(y-1)(y-1-4a) < 0$
	Alternative method for question 3(b)		
	$ax^2 + x - 1 = yx - y$ leading to $ax^2 - (y-1)x + (y-1) = 0$	M1 A1	Forms three term quadratic in x.
	$(y-1)^2 - 4a(y-1) \geq 0$	M1	Correct inequality using discriminant. Must be clear looking for where values of y exist.
	$1 < y < 1 + 4a$	A1	AG Clear method to reach given answer eg $(y-1)(y-1-4a) \geqslant 0$
	Second alternative method for question 3(b)		
	$\frac{dy}{dx} = \frac{(x-1)(2ax+1) - (ax^2 + x - 1)}{(x-1)^2} = 0$	M1	
	$x = 0$ or $x = 2$	A1	
	Proves (0, 1) is local maximum and $(2, 4a+1)$ is local minimum.	M1	Either $\frac{d^2y}{dx^2}$ or gradient either side
	Reference to position of asymptote or two distinct branches to justify $1 < y < 1 + 4a$	A1	AG
		4	

Question	Answer	Marks	Guidance
3(c)		**B1 FT**	Asymptotes identified. FT from part **3a**.
		B1	Two branches in correct position relative to correct asymptotes and of correct shape. Good asymptotic behaviour.
		B1	Lower branch in all four quadrants from correct asymptotes.
		3	

Question	Answer	Marks	Guidance
4(a)	$\sum_{r=1}^{n} e^{rx}\left(e^{2x}-2e^{x}+1\right)=e^{3x}-2e^{2x}+e^{x}$ $+e^{4x}-2e^{3x}+e^{2x}$ $+e^{5x}-2e^{4x}+e^{3x}$ $\vdots$ $+e^{(n+1)x}-2e^{nx}+e^{(n-1)x}$ $+e^{(n+2)x}-2e^{(n+1)x}+e^{nx}$	M1 A1	Shows enough complete terms to make pattern of cancelling clear GP method. $\left(e^{2x}-2e^{x}+1\right)\sum_{r=1}^{n} e^{rx}=\left(e^{2x}-2e^{x}+1\right)e^{x}\dfrac{(e^{x})^{n}-1}{e^{x}-1}$
	$=e^{x}-e^{2x}-e^{(n+1)x}+e^{(n+2)x}$	A1	OE $e^{x}(e^{x}-1)(e^{nx}-1)$
		3	
4(b)	$x<0$	B1	Accept $x \leqslant 0$.
	$e^{nx} \to 0$ as $n \to \infty$ leading to $u_1+u_2+u_3+\ldots=e^{x}-e^{2x}$	M1 A1	Must see $e^{nx} \to 0$ [as $n \to \infty$] agreeing with their set of x
		3	
4(c)	$\sum_{r=1}^{n} \ln u_r=\sum_{r=1}^{n}(rx+\ln(e^{x}-1)^{2})$	M1*	Uses laws of logarithms correctly.
	$\sum_{r=1}^{n} \ln u_r=\frac{1}{2}xn(n+1)+n\ln(e^{x}-1)^{2}$	dM1 A1	Applies $\sum_{r=1}^{n} r=\frac{1}{2}n(n+1)$. AEF for $+n\ln(e^{x}-1)^{2}$
		3	

Question	Answer	Marks	Guidance
5(a)	A shear in the x-direction.	B1	Accept 'shear'.
		1	
5(b)	$\mathbf{A}=\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}$ so true when $n=1$.	B1	States base case.
	Assume that it is true for $n=k$, so $\mathbf{A}^k=\begin{pmatrix}1 & ka\\ 0 & 1\end{pmatrix}$.	B1	States inductive hypothesis.
	Then $\mathbf{A}^{k+1}=\begin{pmatrix}1 & ka\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & a+ak\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & (k+1)a\\ 0 & 1\end{pmatrix}$	M1 A1	Multiplies $\mathbf{A}^k$ with $\mathbf{A}$.
	If it is true for [$n=1$ and] $n=k$ then it is also true for $n=k+1$. Hence, [by induction,] true for all positive integers.	A1	Everything correct and states conclusion.
		5	
5(c)	$\mathbf{A}^n\mathbf{B}=\begin{pmatrix}1 & na\\ 0 & 1\end{pmatrix}\begin{pmatrix}b & b\\ \frac{1}{a} & \frac{1}{a}\end{pmatrix}=\begin{pmatrix}b+n & b+n\\ \frac{1}{a} & \frac{1}{a}\end{pmatrix}$	M1 A1	Uses formula given in part **(b)**.
	$\begin{pmatrix}b+n & b+n\\ \frac{1}{a} & \frac{1}{a}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}(b+n)(x+y)\\ \frac{1}{a}(x+y)\end{pmatrix}$	B1	Transforms $\begin{pmatrix}x\\ y\end{pmatrix}$ by multiplying matrices to find $\begin{pmatrix}X\\ Y\end{pmatrix}$.
	$\frac{1}{a}(1+m)=m(b+n)(1+m)$	M1	Uses $y=mx$ and $Y=mX$
	$y=-x$	A1	
	$y=\frac{1}{a(b+n)}x$	A1	
		6	

Question	Answer	Marks	Guidance
6(a)	$x = r\cos\theta \quad y = r\sin\theta \quad x^2 + y^2 = r^2$	**B1**	SOI
	$r^2\left(1+\sin\theta\cos\theta\right) = r^2\left(1+\frac{1}{2}\sin 2\theta\right) = a$	**M1**	Eliminates both x and y and uses the double angle formula.
	$r^2 = \dfrac{2a}{2+\sin 2\theta}$	**A1**	AG
		3	
6(b)	 	**B1**	Polar graph with curve in correct domain. Graph needs gradient $\leqslant 0$ and $r > 0$ for all θ in the domain.
		B1	r strictly decreasing such that r at $\theta = \dfrac{\pi}{4}$ is greater than half r at $\theta = 0$. Concave graph. Gradient at $\theta = 0$ and $\dfrac{\pi}{4}$ must not be vertical or horizontal, respectively.
		2	

Question	Answer	Marks	Guidance
6(c)	$R = \frac{1}{2}\int_0^{\frac{1}{4}\pi} r^2\, d\theta = a\int_0^{\frac{1}{4}\pi} \frac{1+\tan^2\theta}{2(1+\tan^2\theta)+2\tan\theta}\, d\theta$	B1	AG
	$t = \tan\theta$ leading to $\frac{\mathrm{d}t}{\mathrm{d}\theta} = \sec^2\theta$	M1	Applies given substitution.
	$\frac{\mathrm{d}t}{\mathrm{d}\theta} = \sec^2\theta = 1+t^2$	M1	Applies $\sec^2\theta = 1+\tan^2\theta$
	$R = \frac{1}{2}a\int_0^1 \frac{1}{t^2+t+1}\,\mathrm{d}t$	A1	
	$t^2+t+1 = (t+\frac{1}{2})^2+\frac{3}{4}$	B1	Completes the square.
	$R = \frac{1}{2}a\int_0^1 \frac{1}{(t+\frac{1}{2})^2+\frac{3}{4}}\,\mathrm{d}t = \frac{1}{\sqrt{3}}a\left[\tan^{-1}\left(\frac{2}{\sqrt{3}}t+\frac{1}{\sqrt{3}}\right)\right]_0^1$	M1 A1	Applies formula.
	$\frac{\pi a}{6\sqrt{3}}$	A1	
		8	

Question	Answer	Marks	Guidance
7(a)	$\overrightarrow{AB} = 4\mathbf{i} - \mathbf{j} + \mathbf{k} \quad \overrightarrow{CD} = \mathbf{j} + (\lambda - 3)\mathbf{k}$	B1	Finds direction vectors of the two lines.
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & 1 \\ 0 & 1 & \lambda-3 \end{vmatrix} = \begin{pmatrix} 2-\lambda \\ 12-4\lambda \\ 4 \end{pmatrix}$	M1 A1	Finds common perpendicular. This may also be done by setting up simultaneous equations and solving them.
	$\left\| \dfrac{1}{\sqrt{(\lambda-2)^2 + (4\lambda-12)^2 + 16}} \begin{pmatrix} -5 \\ 2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2-\lambda \\ 12-4\lambda \\ 4 \end{pmatrix} \right\|$	M1 A1	Uses formula for perpendicular distance.
	$\left\| \dfrac{30-3\lambda}{\sqrt{17\lambda^2 - 100\lambda + 164}} \right\| = 3 \Rightarrow 9(\lambda-10)^2 = 9\left(17\lambda^2 - 100\lambda + 164\right)$	M1	Sets equal to 3 and forms in quadratic in λ.
	$16\lambda^2 - 80\lambda + 64 = 0$ leading to $\lambda^2 - 5\lambda + 4 = 0$	A1	AG
		7	
7(b)(i)	$\mathbf{r} = 7\mathbf{i} + 4\mathbf{j} - \mathbf{k} + s\left(4\mathbf{i} - \mathbf{j} + \mathbf{k}\right) + t\left(-5\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}\right)$	M1 A1	OE M1 for using a correct point and attempting to find relevant direction vectors.
		2	
7(b)(ii)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & 1 \\ -5 & 3 & 5 \end{vmatrix} = \begin{pmatrix} -8 \\ -25 \\ 7 \end{pmatrix}$	M1 A1	Finds normal to the plane Π_2.
	$-8(7) - 25(4) + 7(-1)$ leading to $-8x - 25y + 7z = -163$	M1 A1	Substitutes point.
		4	

Question	Answer	Marks	Guidance
7(c)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & 1 \\ -5 & 3 & 2 \end{vmatrix} = \begin{pmatrix} -5 \\ -13 \\ 7 \end{pmatrix}$	M1 A1	Finds normal to the plane Π_1.
	$\begin{pmatrix} -5 \\ -13 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} -8 \\ -25 \\ 7 \end{pmatrix} = \sqrt{243}\sqrt{738}\cos\theta$ leading to $\cos\theta = \dfrac{414}{\sqrt{243}\sqrt{738}}$	M1 A1	Uses dot product of normal vectors. $\cos = 0.9776176...$
	12.1°	A1	Mark final answer. Accept 0.212^c
		5	