Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

6074280448

FURTHER MATHEMATICS **9231/11**

Paper 1 Further Pure Mathematics 1 **October/November 2021**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Any blank pages are indicated.

DC (PQ) 199017/3

1 It is given that

$$\alpha+\beta+\gamma = 3, \quad \alpha^2+\beta^2+\gamma^2 = 5, \quad \alpha^3+\beta^3+\gamma^3 = 6.$$

The cubic equation $x^3+bx^2+cx+d=0$ has roots α, β, γ.

Find the values of b, c and d. [6]

2 **(a)** Use standard results from the list of formulae (MF19) to find $\sum_{r=1}^{n} r(r+1)(r+2)$ in terms of n, fully factorising your answer. [3]

(b) Express $\frac{1}{r(r+1)(r+2)}$ in partial fractions and hence use the method of differences to find

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}.$$ [5]

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(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)}$. [1]

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3 The sequence of real numbers $a_1, a_2, a_3, \dots$ is such that $a_1 = 1$ and

$$a_{n+1} = \left(a_n + \frac{1}{a_n}\right)^3.$$

(a) Prove by mathematical induction that $\ln a_n \geqslant 3^{n-1} \ln 2$ for all integers $n \geqslant 2$. [6]

[You may use the fact that $\ln\left(x + \frac{1}{x}\right) > \ln x$ for $x > 0$.]

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(b) Show that $\ln a_{n+1} - \ln a_n > 3^{n-1} \ln 4$ for $n \geqslant 2$. [2]

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4 The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.

(a) The matrix $\mathbf{M}$ represents a sequence of two geometrical transformations.

State the type of each transformation, and make clear the order in which they are applied. [2]

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(b) Find the values of θ, for $0 \leqslant \theta \leqslant \pi$, for which the transformation represented by $\mathbf{M}$ has exactly one invariant line through the origin, giving your answers in terms of π. [9]

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5 The plane Π has equation $\mathbf{r} = -2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} + \mathbf{k}) + \mu(2\mathbf{i} + 3\mathbf{j})$.

(a) Find a Cartesian equation of Π, giving your answer in the form $ax + by + cz = d$. [4]

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The line l passes through the point P with position vector $2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ and is parallel to the vector $\mathbf{k}$.

(b) Find the position vector of the point where l meets Π. [3]

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(c) Find the acute angle between l and Π. [3]

(d) Find the perpendicular distance from P to Π. [3]

6 The curve C has polar equation $r = 2\cos\theta(1 + \sin\theta)$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) Find the polar coordinates of the point on C that is furthest from the pole. [5]

(b) Sketch C. [2]

(c) Find the area of the region bounded by C and the initial line, giving your answer in exact form. [6]

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7 The curve C has equation $y = \dfrac{4x+5}{4-4x^2}$.

(a) Find the equations of the asymptotes of C. [2]

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(b) Find the coordinates of any stationary points on C. [4]

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(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left|\dfrac{4x+5}{4-4x^2}\right|$ and find in exact form the set of values of x for which $4|4x+5| > 5|4-4x^2|$. [6]

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Additional Page

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MARK SCHEME

Maximum Mark: 75

Published

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This document consists of **14** printed pages.

Question	Answer	Marks	Guidance
1	$b=-(\alpha+\beta+\gamma)=-3$	B1	
	$5=3^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$	M1 A1	Uses formula for sum of squares.
	$c=2$	A1	
	$6-3(5)+2(3)+3d=0$	M1	Uses original equation or formula for sum of cubes.
	[Equation is $x^3-3x^2+2x+1=0$] $d=1$	A1	
		6	

Question	Answer	Marks	Guidance
2(a)	$\sum_{r=1}^{n}\left(r^3+3r^2+2r\right)=\frac{1}{4}n^2(n+1)^2+\frac{1}{2}n(n+1)(2n+1)+n(n+1)$	M1 A1	Expands and substitutes formulae. At least two formulae used for M1.
	$=\frac{1}{4}n(n+1)\left(n^2+n+4n+2+4\right)=\frac{1}{4}n(n+1)(n+2)(n+3)$	A1	
		3	

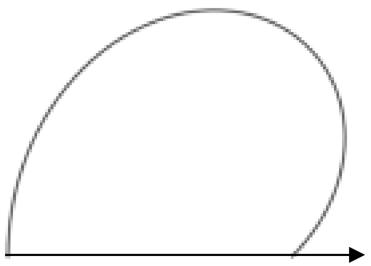
Question	Answer	Marks	Guidance
2(b)	$\frac{1}{r(r+1)(r+2)}=\frac{1}{2r}-\frac{1}{r+1}+\frac{1}{2(r+2)}$	M1 A1	Finds partial fractions.
	$\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}=\ \frac{1}{2}f(1)-f(2)+\frac{1}{2}f(3)$ $+\frac{1}{2}f(2)-f(3)+\frac{1}{2}f(4)$ $+\frac{1}{2}f(3)-f(4)+\frac{1}{2}f(5)$ $\vdots$ $+\frac{1}{2}f(n-1)-f(n)+\frac{1}{2}f(n+1)$ $+\frac{1}{2}f(n)-f(n+1)+\frac{1}{2}f(n+2)$	M1 A1	Shows enough terms for cancelation to be clear. Where $f(x)=\frac{1}{r}$
	$=\frac{1}{4}-\frac{1}{2(n+1)}+\frac{1}{2(n+2)}$	A1	ISW
		5	
2(c)	$\frac{1}{4}$	B1 FT	FT from *their* answer to part **(b)**.
		1	

Question	Answer	Marks	Guidance
3(a)	$a_2 = 2^3$ leading to $\ln a_2 = 3\ln 2$ so true when $n = 2$.	B1	Checks base case. Must be n =2 and exact.
	Assume that $\ln a_k \geqslant 3^{k-1}\ln 2$.	B1	States inductive hypothesis.
	$\ln a_{k+1} = 3\ln\left(a_k + \frac{1}{a_k}\right) > 3\ln a_k$	M1	Applies result given in part **(a)**.
	$\geqslant 3^k \ln 2$	M1 A1	Applies inductive hypothesis and writes in required form.
	So true when $n = k+1$. By induction, true for all integers $n \geqslant 2$.	A1	States conclusion, all previous marks must be gained.
		6	
3(b)	$3\ln\left(a_n + \frac{1}{a_n}\right) - \ln a_n > 2\ln a_n$	M1	Applies results given in parts **(a)** and **(b)**.
	$> 2\times 3^{n-1}\ln 2 = 3^{n-1}\ln 4$	A1	AG
		2	

Question	Answer	Marks	Guidance
4(a)	One-way stretch [scale factor 3 parallel to the x-axis] followed by a rotation, [anticlockwise centred at the origin, through an angle θ.]	B2	Award B1 if given in the wrong order.
		2	

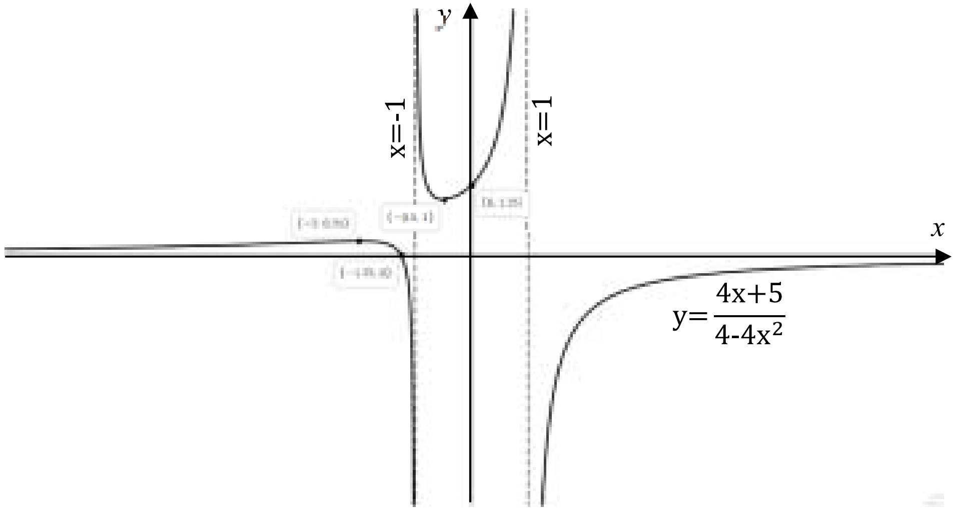
Question	Answer	Marks	Guidance
4(b)	$\mathbf{M}=\begin{pmatrix}3\cos\theta & -\sin\theta\\ 3\sin\theta & \cos\theta\end{pmatrix}$	B1	
	$\begin{pmatrix}3\cos\theta & -\sin\theta\\ 3\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}3x\cos\theta-y\sin\theta\\ 3x\sin\theta+y\cos\theta\end{pmatrix}$	B1	Transforms $\begin{pmatrix}x\\ y\end{pmatrix}$ to $\begin{pmatrix}X\\ Y\end{pmatrix}$.
	$3x\sin\theta+mx\cos\theta=3mx\cos\theta-m^2x\sin\theta$	M1 A1	Uses $y=mx$ and $Y=mX$.
	$m^2\sin\theta-2m\cos\theta+3\sin\theta=0$	A1	
	$4\cos^2\theta-12\sin^2\theta=0$	M1 A1	Sets discriminant equal to 0.
	$\tan\theta=\pm\frac{1}{\sqrt{3}}$ or $4\cos^2\theta-12\left(1-\cos^2\theta\right)=0$ leading to $16\cos^2\theta-12=0$ leading to $\cos\theta=\pm\frac{\sqrt{3}}{2}$	M1	Uses an appropriate trigonometric identity.
	$\theta=\frac{1}{6}\pi$ and $\theta=\frac{5}{6}\pi$	A1	Allow 30° and 150°
		9	

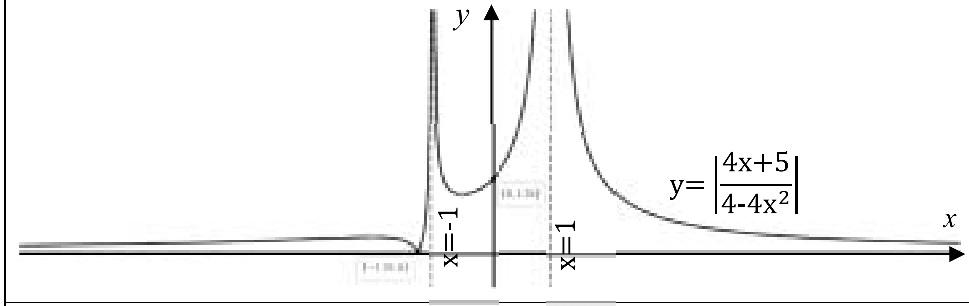
Question	Answer	Marks	Guidance
5(a)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 2 & 3 & 0 \end{vmatrix} = \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix}$	M1 A1	Finds vector perpendicular to Π.
	$-3(-2)+2(3)+3(3)=21$	M1	Substitutes point on Π.
	$-3x+2y+3z=21$	A1	
		4	
5(b)	$\begin{pmatrix} 2 \\ -3 \\ 5+t \end{pmatrix}$	B1	Forms general point on line (given as a single vector).
	$-3(2)+2(-3)+3(5+t)=21$ leading to $t=6$	M1	Substitutes into the equation for Π.
	$\begin{pmatrix} 2 \\ -3 \\ 11 \end{pmatrix}$	A1	
		3	
5(c)	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix} = \sqrt{1}\sqrt{22}\cos\alpha$ leading to $\cos\alpha = \frac{3}{\sqrt{22}}$	M1 A1 FT	Uses dot product of $\mathbf{k}$ and their normal.
	Acute angle between l and Π is $90-\alpha=39.8^{\circ}$	A1	
		3	

Question	Answer	Marks	Guidance
5(d)	$\begin{pmatrix}0\\0\\7\end{pmatrix}-\begin{pmatrix}2\\-3\\5\end{pmatrix}=\begin{pmatrix}-2\\3\\2\end{pmatrix}$	B1	Finds direction vector from P to plane.
	$\frac{1}{\sqrt{22}}\begin{pmatrix}-2\\3\\2\end{pmatrix}\cdot\begin{pmatrix}-3\\2\\3\end{pmatrix}=\frac{18}{\sqrt{22}}=3.84$	M1 A1	Uses dot product of their direction and normal vectors.
		3	
6(a)	$2\left(c(c)-(1+s)s\right)=0$	M1	Finds $\frac{dr}{d\theta}$ and sets equal to 0.
	$1-s-2s^2=0$	A1	
	$\sin\theta=-1$, $\sin\theta=\frac{1}{2}$ leading to $\theta=\frac{1}{6}\pi$	M1 A1	Solves quadratic in $\sin\theta$.
	$\left(\frac{3}{2}\sqrt{3},\frac{1}{6}\pi\right)$	A1	Allow $\left(2.60,\frac{1}{6}\pi\right)$ 3sf
		5	
6(b)		B1	Correct shape. Tangential to $\theta=\frac{\pi}{2}$ at pole, r strictly increasing to $\frac{\pi}{6}$ then decreasing.
		B1	Correct position. All in first quadrant.
		2	

Question	Answer	Marks	Guidance
6(c)	$\frac{1}{2}\int_0^{\frac{1}{2}\pi} 4c^2(1+s)^2\ \mathrm{d}\theta = 2\int_0^{\frac{1}{2}\pi} c^2+2sc^2+s^2c^2\ \mathrm{d}\theta$	M1 A1	Uses $\frac{1}{2}\int r^2\ \mathrm{d}\theta$ with correct limits.
	$2\int_0^{\frac{1}{2}\pi}\frac{1}{2}(\cos 2\theta+1)+2\sin\theta\cos^2\theta+\frac{1}{4}\sin^2 2\theta\ \mathrm{d}\theta$	M1	Uses double angle formulae on first and last terms.
	$\int_0^{\frac{1}{2}\pi}\cos 2\theta+1+4\sin\theta\cos^2\theta+\frac{1}{4}-\frac{1}{4}\cos 4\theta\ \mathrm{d}\theta$	A1	Obtains integrable form.
	$\left[\frac{5}{4}\theta+\frac{1}{2}\sin 2\theta-\frac{4}{3}\cos^3\theta-\frac{1}{16}\sin 4\theta\right]_0^{\frac{1}{2}\pi}$	M1	Integrates.
	$\frac{5}{8}\pi+\frac{4}{3}$	A1	
		6	

Question	Answer	Marks	Guidance
7(a)	$x=1,\ \ x=-1$	B1	Vertical asymptotes.
	$y=0$	B1	Horizontal asymptote.
		2	
7(b)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(4-4x^2)(4)-(4x+5)(-8x)}{\left(4-4x^2\right)^2}$	M1	Finds $\frac{\mathrm{d}y}{\mathrm{d}x}$.
	$16x^2+40x+16=0$	M1	Sets equal to 0 and forms equation.
	$\left(-2,\frac{1}{4}\right),\ \ \left(-\frac{1}{2},1\right)$	A1 A1	
		4	

Question	Answer	Marks	Guidance
7(c)		B1	Axes and asymptotes.
		B1	Correct shape and position.
	$(-\frac{5}{4},0)$, $(0,\frac{5}{4})$	B1	States coordinates of intersections with axes.
		3	

Question	Answer	Marks	Guidance
7(d)		B1 FT	FT from sketch in part **(c).**
		B1	Correct shape at infinity.
	$\frac{4x+5}{4-4x^2}=\frac{5}{4}$ or $\frac{4x+5}{4-4x^2}=-\frac{5}{4}$ $5x^2+4x=0$ or $5x^2-4x-10=0$	M2	Finds critical points, award M1 for each case.
	$x=-\frac{4}{5},\quad x=0$ or $x=\frac{2}{5}-\frac{3}{5}\sqrt{6},\quad x=\frac{2}{5}+\frac{3}{5}\sqrt{6}$	A1	A0 if $-1.07,\ 1.87$
	$\frac{2}{5}-\frac{3}{5}\sqrt{6}<x<-\frac{4}{5},\quad 0<x<\frac{2}{5}+\frac{3}{5}\sqrt{6}\ ,\quad x\neq\pm1$	A1 FT	Condone exclusion of $x=\pm1$ from the range.
		6	

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Cambridge International AS & A Level

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CENTRE NUMBER

CANDIDATE NUMBER

5983925337

FURTHER MATHEMATICS **9231/12**

Paper 1 Further Pure Mathematics 1 **October/November 2021**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
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- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Any blank pages are indicated.

1 **(a)** Give full details of the geometrical transformation in the *x*-*y* plane represented by the matrix $\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$. [1]

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Let $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 2 & 2 \end{pmatrix}$.

(b) The triangle *DEF* in the *x*-*y* plane is transformed by **A** onto triangle *PQR*.

Given that the area of triangle *DEF* is $13\,\text{cm}^2$, find the area of triangle *PQR*. [2]

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(c) Find the matrix **B** such that $\mathbf{AB} = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$. [2]

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(d) Show that the origin is the only invariant point of the transformation in the *x*-*y* plane represented by **A**. [4]

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2 It is given that $y = xe^{ax}$, where a is a constant.

Prove by mathematical induction that, for all positive integers n,

$$\frac{d^n y}{dx^n} = \left(a^n x + na^{n-1}\right)e^{ax}.$$ [6]

3 Let $S_n = \sum_{r=1}^{n} \ln \frac{r(r+2)}{(r+1)^2}$.

(a) Using the method of differences, or otherwise, show that $S_n = \ln \frac{n+2}{2(n+1)}$. [4]

Let $S = \sum_{r=1}^{\infty} \ln \frac{r(r+2)}{(r+1)^2}$.

(b) Find the least value of n such that $S_n - S < 0.01$. [3]

4 The cubic equation $x^3+2x^2+3x+3=0$ has roots α, β, γ.

(a) Find the value of $\alpha^2+\beta^2+\gamma^2$. [2]

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(b) Show that $\alpha^3+\beta^3+\gamma^3=1$. [2]

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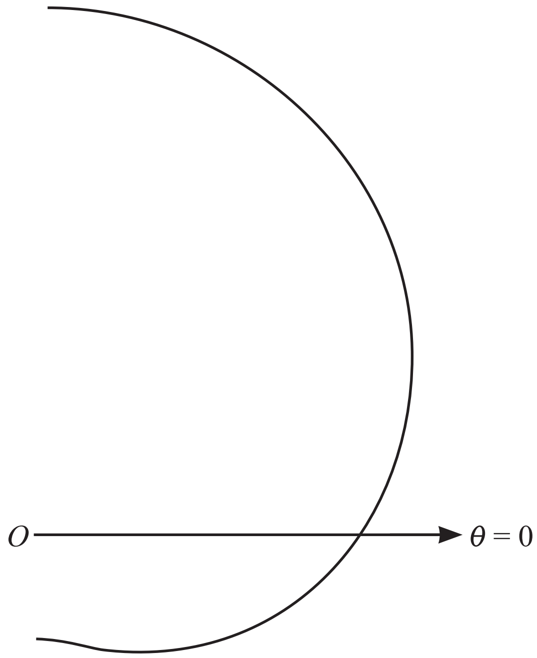
(c) Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{n}\left((\alpha+r)^3+(\beta+r)^3+(\gamma+r)^3\right)=n+\tfrac{1}{4}n(n+1)\left(an^2+bn+c\right),$$

where a, b and c are constants to be determined. [6]

5 The curve C has polar equation $r = 3 + 2\sin\theta$, for $-\pi < \theta \leqslant \pi$.

(a) The diagram shows part of C. Sketch the rest of C on the diagram. [1]



The straight line l has polar equation $r\sin\theta = 2$.

(b) Add l to the diagram in part **(a)** and find the polar coordinates of the points of intersection of C and l. [5]

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(c) The region R is enclosed by C and l, and contains the pole.

Find the area of R, giving your answer in exact form. [6]

6 The curve C has equation $y = \dfrac{x^2}{x-3}$.

(a) Find the equations of the asymptotes of C. [3]

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(b) Show that there is no point on C for which $0 < y < 12$. [4]

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(c) Sketch C. [2]

(d) **(i)** Sketch the graphs of $y = \left|\frac{x^2}{x-3}\right|$ and $y = |x| - 3$ on a single diagram, stating the coordinates of the intersections with the axes. [4]

(ii) Use your sketch to find the set of values of c for which $\left|\frac{x^2}{x-3}\right| \leqslant |x| + c$ has no solution. [1]

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7 The points A, B, C have position vectors

$$2\mathbf{i}+2\mathbf{j}, \quad -\mathbf{j}+\mathbf{k} \quad \text{and} \quad 2\mathbf{i}+\mathbf{j}-7\mathbf{k}$$

respectively, relative to the origin O.

(a) Find an equation of the plane OAB, giving your answer in the form $\mathbf{r.n} = p$. [3]

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The plane Π has equation $x-3y-2z = 1$.

(b) Find the perpendicular distance of Π from the origin. [1]

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(c) Find the acute angle between the planes OAB and Π. [3]

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(d) Find an equation for the common perpendicular to the lines OC and AB. [10]

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Additional Page

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1(a)	Enlargement, scale factor 6.	B1	
		1	
1(b)	$\det \mathbf{A} = 6 - 8 = -2$	M1	Finds $\det \mathbf{A}$.
	Area $= 2 \times 13 = 26\text{ cm}^2$	A1	
		2	
1(c)	$\mathbf{A}^{-1} = -\frac{1}{2}\begin{pmatrix} 2 & -4 \\ -2 & 3 \end{pmatrix}$	M1	Finds $\mathbf{A}^{-1}$.
	$\mathbf{B} = -3\begin{pmatrix} 2 & -4 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -6 & 12 \\ 6 & -9 \end{pmatrix}$	A1	AEF Could be solved by equations.
		2	
1(d)	$\begin{pmatrix} 3 & 4 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x+4y \\ 2x+2y \end{pmatrix}$	B1	Finds $\begin{pmatrix} X \\ Y \end{pmatrix}$
	$\left.\begin{matrix} 3x+4y=x \\ 2x+2y=y \end{matrix}\right\}$ leading to $\left\{\begin{matrix} 2x+4y=0 \\ 2x+y=0 \end{matrix}\right.$	M1	Uses $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$ to form simultaneous equations.
	$y = -2x$ leading to $-6x = 0$ leading to $x = 0,\ y = 0$	M1 A1	Solves equations or states that $\det\begin{pmatrix} 2 & 4 \\ 2 & 1 \end{pmatrix} \neq 0$, AG.
		4	

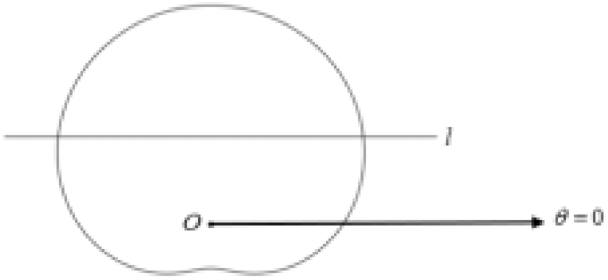
Question	Answer	Marks	Guidance
2	$\frac{\mathrm{d}y}{\mathrm{d}x} = axe^{ax} + e^{ax} = (ax+1)e^{ax}$ so true when $n = 1$.	M1 A1	Differentiates once using the product rule.
	Assume that $\frac{\mathrm{d}^k y}{\mathrm{d}x^k} = \left(a^k x + ka^{k-1}\right)e^{ax}$.	B1	States inductive hypothesis.
	$\frac{\mathrm{d}^{k+1} y}{\mathrm{d}x^{k+1}} = a\left(a^k x + ka^{k-1}\right)e^{ax} + e^{ax}\left(a^k\right) = \left(a^{k+1} + (k+1)a^k\right)e^{ax}$	M1 A1	Differentiates kth derivative.
	So true when $n = k+1$. By induction, true for all positive integers n.	A1	States conclusion.
		6	

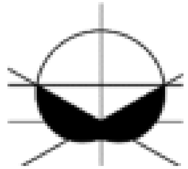
Question	Answer	Marks	Guidance
3(a)	$\sum_{r=1}^{n}\left(\ln r - 2\ln(r+1) + \ln(r+2)\right)$	B1	Separates logarithms into correct form using a difference. Or as logarithm of product.
	$\ln 1 - 2\ln 2 + \ln 3$ $\ln 2 - 2\ln 3 + \ln 4$ $\ln 3 - 2\ln 4 + \ln 5$ $\vdots$ $\ln(n-1) - 2\ln n + \ln(n+1)$ $\ln n - 2\ln(n+1) + \ln(n+2)$	M1 A1	Shows enough terms to make cancellation clear.
	$[\ln 1] - \ln 2 - \ln(n+1) + \ln(n+2) = \ln\frac{n+2}{2(n+1)}$.	A1	AG
		4	

Question	Answer	Marks	Guidance
3(b)	$S = -\ln 2$	B1	States sum to infinity. AEF
	$S_n - S = \ln\left(\dfrac{n+2}{n+1}\right) < 0.01$ leading to $n+2 < e^{0.01}(n+1)$	M1	Forms inequality
	Least value of n is 99	A1	CAO
		3	

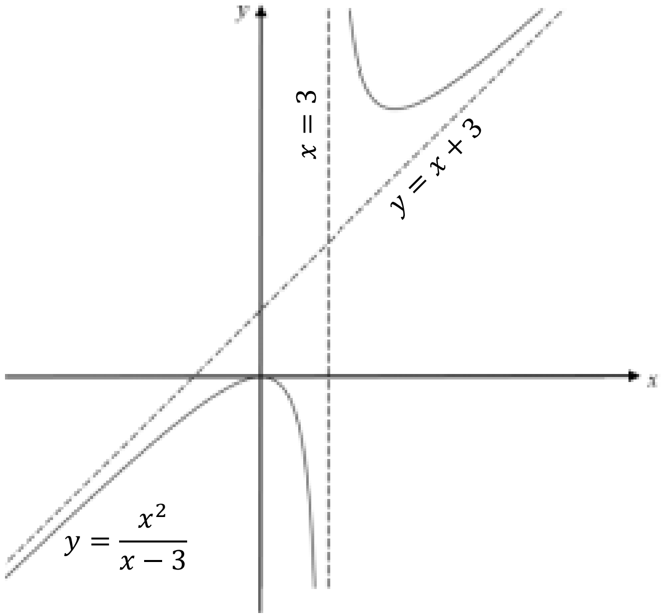
Question	Answer	Marks	Guidance
4(a)	$(-2)^2 - 2(3)$	M1	Uses formula for sum of squares.
	-2	A1	
		2	
4(b)	$\alpha^3 + \beta^3 + \gamma^3 = -2(-2) - 3(-2) - 3(3)$	M1	Uses original equation or formula for sum of cubes.
	1	A1	AG
		2	

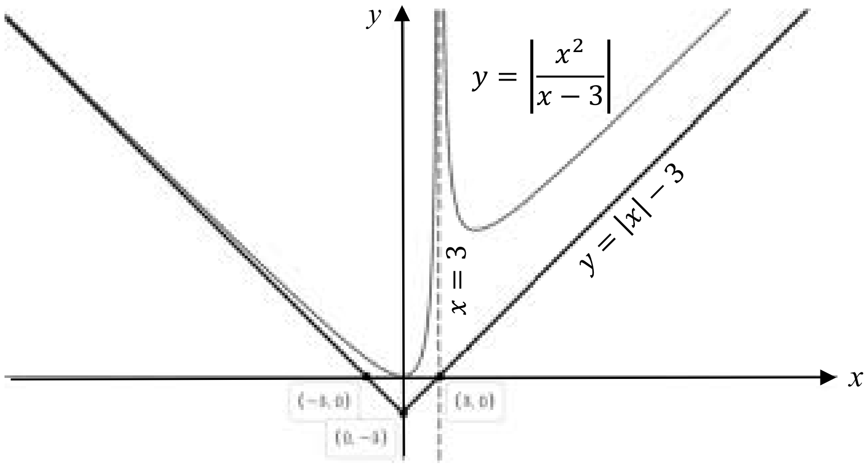
Question	Answer	Marks	Guidance
4(c)	$(\alpha+r)^3=\alpha^3+3\alpha^2 r+3\alpha r^2+r^3$	**B1**	Expands.
	$\sum_{r=1}^{n}\left((\alpha+r)^3+(\beta+r)^3+(\gamma+r)^3\right)=\sum_{r=1}^{n}\left(1+3(-2)r+3(-2)r^2+3r^3\right)$	**M1 A1**	Collects like terms and uses results from parts **(a)** and **(b)**.
	$n-6\left(\frac{1}{2}n(n+1)\right)-6\left(\frac{1}{6}n(n+1)(2n+1)\right)+\frac{3}{4}n^2(n+1)^2$ $n-3n(n+1)-n(n+1)(2n+1)+\frac{3}{4}n^2(n+1)^2$	**M1**	Applies formulae from MF19.
	$n+\frac{1}{4}n(n+1)\left(-12-4(2n+1)+3n(n+1)\right)$ $n+\frac{1}{4}n(n+1)\left(3n^2-5n-16\right)$	**M1 A1**	Simplifies.
		6	

Question	Answer	Marks	Guidance
5(a)		B1	Correct symmetrical shape, closed loop.
		1	
5(b)	Line l parallel to initial line and correct side of pole.	B1	
	$2 = 3\sin\theta + 2\sin^2\theta$	M1	Forms quadratic in $\sin\theta$. Or in r $r = 3 + \frac{4}{r}$
	$\sin\theta = \frac{1}{2}$	M1	Solves for $\sin\theta$.
	$\left(4, \frac{1}{6}\pi\right)$ $\left(4, \frac{5}{6}\pi\right)$	A1 A1	**SC1** For finding both angles correctly
		5	

Question	Answer	Marks	Guidance
5(c)	$2\times\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}}(3+2\sin\theta)^2\,\mathrm{d}\theta$	M1	Finds the part of the required area enclosed by the curved outer edge and two line segments from the pole. Limits must be correct. 
	$\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}}9+12\sin\theta+2(1-\cos 2\theta)\,\mathrm{d}\theta$	M1	Uses double angle formula and integrates.
	$[11\theta-12\cos\theta-\sin 2\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{6}}=\frac{22}{3}\pi-\frac{13}{2}\sqrt{3}$	A1 A1	
	$\frac{22}{3}\pi-\frac{13}{2}\sqrt{3}+(4\cos\frac{\pi}{6})\times 2$	M1	Adds area of triangle.
	$\frac{22}{3}\pi-\frac{5}{2}\sqrt{3}$	A1	
		6	

Question	Answer	Marks	Guidance
6(a)	$x = 3$	**B1**	States vertical asymptote.
	$y = x + 3 + \dfrac{9}{x-3}$ leading to $y = x + 3$	**M1 A1**	Finds oblique asymptote.
		3	
6(b)	$yx - 3y = x^2$ leading to $x^2 - yx + 3y = 0$	**M1 A1**	Forms quadratic in x.
	$y^2 - 4(3y) < 0$ leading to $y^2 - 12y < 0$	**M1**	Uses that discriminant is negative.
	$0 < y < 12$	**A1**	AG
		4	

Question	Answer	Marks	Guidance
6(c)		B1	Axes and asymptotes.
		B1	Branches correct.
		2	

Question	Answer	Marks	Guidance
6(d)(i)		B1 FT	FT from sketch in **(c)**.
		B1	Correct shape at infinity.
		B1	Correct shape of $y = \|x\| - 3$.
		B1	Correct intercepts with axes (may be seen on graph).
		4	
6(d)(ii)	$c \leqslant -3$	B1	
		1	

Question	Answer	Marks	Guidance
7(a)	$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 0 & -1 & 1 \end{vmatrix} = \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix} \sim \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$	**M1 A1**	Finds common perpendicular.
	$\mathbf{r}.\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = 0$	**A1**	
		3	
7(b)	$\frac{1}{\sqrt{1^2+3^2+2^2}} = \frac{1}{\sqrt{14}}$	**B1**	Divides by magnitude of the normal to Π 0.267
		1	
7(c)	$\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} = \sqrt{3}\sqrt{14}\cos\alpha$ leading to $\cos\alpha = \frac{6}{\sqrt{3}\sqrt{14}}$	**M1** **A1 FT**	Takes dot product of normal vectors.
	22.2°	**A1**	Accept 0.388 radians. Mark final answer.
		3	

Question	Answer	Marks	Guidance
7(d)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -7 \\ -2 & -3 & 1 \end{vmatrix} = \begin{pmatrix} -20 \\ 12 \\ -4 \end{pmatrix} \sim \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}$	M1 A1	Finds direction of common perpendicular.
	$\overrightarrow{OP} = \lambda \begin{pmatrix} 2 \\ 1 \\ -7 \end{pmatrix}, \overrightarrow{OQ} = \begin{pmatrix} 2-2\mu \\ 2-3\mu \\ \mu \end{pmatrix}$ leading to $\overrightarrow{PQ} = \begin{pmatrix} 2-2\mu-2\lambda \\ 2-3\mu-\lambda \\ \mu+7\lambda \end{pmatrix}$	M1 A1	Finds $\overrightarrow{PQ}$.
	$\begin{pmatrix} 2-2\mu-2\lambda \\ 2-3\mu-\lambda \\ \mu+7\lambda \end{pmatrix} \bullet \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix} = 0$ or $\begin{pmatrix} 2-2\mu-2\lambda \\ 2-3\mu-\lambda \\ \mu+7\lambda \end{pmatrix} = k \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}$	M1	Uses that dot product of $\overrightarrow{PQ}$ with line direction is zero, or, alternatively, $\overrightarrow{PQ}$ is a multiple of the common perpendicular (parameter k not 1).
	$14\mu + 14\lambda = 10$	A1	Deduces one equation.
	$\begin{pmatrix} 2-2\mu-2\lambda \\ 2-3\mu-\lambda \\ \mu+7\lambda \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ -7 \end{pmatrix} = 0 \Rightarrow 14\mu + 54\lambda = 6$	A1	Deduces second equation.
	$\lambda = -\frac{1}{10}$ leading to $\overrightarrow{OP} = -\frac{1}{10}\begin{pmatrix} 2 \\ 1 \\ -7 \end{pmatrix}$	M1 A1	Solves for λ and substitutes into $\overrightarrow{OP}$.
	$\mathbf{r} = \begin{pmatrix} -0.2 \\ -0.1 \\ 0.7 \end{pmatrix} + k \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}$	B1 FT	FT using their common perpendicular.
		10	

Cambridge Assessment International Education

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FURTHER MATHEMATICS **9231/11**

Paper 1 Further Pure Mathematics 1 **May/June 2021**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Any blank pages are indicated.

DC (LO) 214849

1 Prove by mathematical induction that $2^{4n} + 31^n - 2$ is divisible by 15 for all positive integers n. [6]

2 **(a)** Use standard results from the List of formulae (MF19) to find $\sum_{r=1}^{n}\left(1-r-r^2\right)$ in terms of n, simplifying your answer. [3]

(b) Show that

$$\frac{1-r-r^2}{(r^2+2r+2)(r^2+1)} = \frac{r+1}{(r+1)^2+1} - \frac{r}{r^2+1}$$

and hence use the method of differences to find $\sum_{r=1}^{n} \frac{1-r-r^2}{(r^2+2r+2)(r^2+1)}$. [5]

(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1-r-r^2}{(r^2+2r+2)(r^2+1)}$. [1]

3 The equation $x^4 - 2x^3 - 1 = 0$ has roots α, β, γ, δ.

(a) Find a quartic equation whose roots are α^3, β^3, γ^3, δ^3 and state the value of $\alpha^3 + \beta^3 + \gamma^3 + \delta^3$. [4]

(b) Find the value of $\frac{1}{\alpha^3}+\frac{1}{\beta^3}+\frac{1}{\gamma^3}+\frac{1}{\delta^3}$. [3]

(c) Find the value of $\alpha^4+\beta^4+\gamma^4+\delta^4$. [2]

4 The matrix **M** represents the sequence of two transformations in the x-y plane given by a rotation of 60° anticlockwise about the origin followed by a one-way stretch in the x-direction, scale factor d ($d \neq 0$).

(a) Find **M** in terms of d. [4]

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(b) The unit square in the x-y plane is transformed by **M** onto a parallelogram of area $\frac{1}{2}d^2$ units^2.

Show that $d = 2$. [2]

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The matrix **N** is such that $\mathbf{MN} = \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

(c) Find **N**. [3]

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(d) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **MN**. [5]

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5 The curve C has polar equation $r = a\cot\left(\frac{1}{3}\pi - \theta\right)$, where a is a positive constant and $0 \leqslant \theta \leqslant \frac{1}{6}\pi$.

It is given that the greatest distance of a point on C from the pole is $2\sqrt{3}$.

(a) Sketch C and show that $a = 2$. [3]

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(b) Find the exact value of the area of the region bounded by C, the initial line and the half-line $\theta = \frac{1}{6}\pi$. [4]

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(c) Show that C has Cartesian equation $2\left(x+y\sqrt{3}\right)=\left(x\sqrt{3}-y\right)\sqrt{x^2+y^2}$. [3]

6 Let t be a positive constant.

The line l_1 passes through the point with position vector $t\mathbf{i}+\mathbf{j}$ and is parallel to the vector $-2\mathbf{i}-\mathbf{j}$. The line l_2 passes through the point with position vector $\mathbf{j}+t\mathbf{k}$ and is parallel to the vector $-2\mathbf{j}+\mathbf{k}$.

It is given that the shortest distance between the lines l_1 and l_2 is $\sqrt{21}$.

(a) Find the value of t. [5]

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The plane Π_1 contains l_1 and is parallel to l_2.

(b) Write down an equation of Π_1, giving your answer in the form $\mathbf{r} = \mathbf{a}+\lambda\mathbf{b}+\mu\mathbf{c}$. [1]

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The plane Π_2 has Cartesian equation $5x - 6y + 7z = 0$.

(c) Find the acute angle between l_2 and Π_2. [3]

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(d) Find the acute angle between Π_1 and Π_2. [3]

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7 The curve C has equation $y = \frac{x^2 + x + 9}{x + 1}$.

(a) Find the equations of the asymptotes of C. [3]

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(b) Find the coordinates of the stationary points on C. [4]

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(c) Sketch C, stating the coordinates of any intersections with the axes. [3]

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(d) Sketch the curve with equation $y = \left|\dfrac{x^2+x+9}{x+1}\right|$ and find the set of values of x for which $2\left|x^2+x+9\right| > 13\left|x+1\right|$. [5]

Cambridge Assessment International Education

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FURTHER MATHEMATICS **9231/11**

Paper 1 Further Pure Mathematics 1 **May/June 2021**

MARK SCHEME

Maximum Mark: 75

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the May/June 2021 series for most Cambridge IGCSE™, Cambridge International A and AS Level components and some Cambridge O Level components.

This document consists of **14** printed pages.

Question	Answer	Marks	Guidance
1	$2^4+31-2=45$ is divisible by 15	**B1**	Checks base case.
	Assume that $2^{4k}+31^k-2$ is divisible by 15 for some positive integer *k*.	**B1**	States inductive hypothesis.
	Then $2^{4k+4}+31^{k+1}-2=(15+1)2^{4k}+(30+1)31^k-2$	**M1 A1**	Separates $2^{4k}+31^k-2$ or considers difference.
	is divisible by 15 because $15\times2^{4k}+30\times31^k$ is divisible by 15.	**A1**	
	Hence, by induction, true for every positive integer n.	**A1**	
		6	

Question	Answer	Marks	Guidance
2(a)	$n-\frac{1}{2}n(n+1)-\frac{1}{6}n(n+1)(2n+1)$	**M1A1**	Substitutes correct formulae from MF19. Expanding brackets correctly.
	$\frac{1}{3}n-n^2-\frac{1}{3}n^3$	**A1**	Simplifies by collecting terms.
		3	

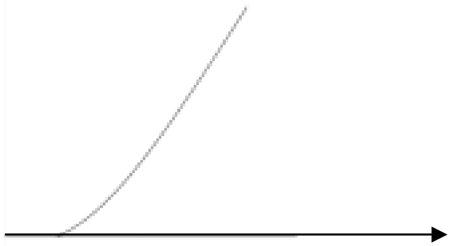
Question	Answer	Marks	Guidance
2(b)	$\dfrac{r+1}{(r+1)^2+1}-\dfrac{r}{r^2+1}=\dfrac{(r+1)\left(r^2+1\right)-r\left(r^2+2r+2\right)}{\left(r^2+2r+2\right)\left(r^2+1\right)}=\dfrac{1-r-r^2}{\left(r^2+2r+2\right)\left(r^2+1\right)}$	M1 A1	Puts over a common denominator and expands, AG.
	$\sum_{r=1}^{n}\dfrac{1-r-r^2}{\left(r^2+2r+2\right)\left(r^2+1\right)}=\sum_{r=1}^{n}\left(\dfrac{r+1}{(r+1)^2+1}-\dfrac{r}{r^2+1}\right)$ $=\dfrac{2}{5}-\dfrac{1}{2}+\dfrac{3}{10}-\dfrac{2}{5}+\dfrac{4}{17}-\dfrac{3}{10}+\cdots+\dfrac{n+1}{(n+1)^2+1}-\dfrac{n}{n^2+1}$	M1 A1	Shows at least three complete terms including first and last. Cancellation may be implicit.
	$=-\dfrac{1}{2}+\dfrac{n+1}{(n+1)^2+1}$	A1	ISW
		5	
2(c)	$-\frac{1}{2}$	B1 FT	FT from their answer to part **(b)**.
		1	

Question	Answer	Marks	Guidance
3(a)	$y=x^3$	B1	Correct substitution.
	$y^{\frac{4}{3}}-2y-1=0\Rightarrow y^4=(2y+1)^3=8y^3+12y^2+6y+1$	M1	Obtains an equation not involving radicals.
	$y^4-8y^3-12y^2-6y-1=0$	A1	
	$\alpha^3+\beta^3+\gamma^3+\delta^3=8$	B1 FT	
		4	

Question	Answer	Marks	Guidance
3(b)	$\frac{1}{\alpha^3}+\frac{1}{\beta^3}+\frac{1}{\gamma^3}+\frac{1}{\delta^3}=\frac{\alpha^3\beta^3\delta^3+\alpha^3\beta^3\gamma^3+\beta^3\gamma^3\delta^3+\alpha^3\gamma^3\delta^3}{\alpha^3\beta^3\gamma^3\delta^3}=\frac{6}{-1}$	M1 A1 FT	Relates $\frac{1}{\alpha^3}+\frac{1}{\beta^3}+\frac{1}{\gamma^3}+\frac{1}{\delta^3}$ to coefficients.
	-6	A1	
		3	
3(c)	$\alpha^4+\beta^4+\gamma^4+\delta^4=2(\alpha^3+\beta^3+\gamma^3+\delta^3)+4$	M1	Uses original equation.
	$=20$	A1	
		2	

Question	Answer	Marks	Guidance
4(a)	$\begin{pmatrix}\cos 60 & -\sin 60\\ \sin 60 & \cos 60\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}$	B1	Rotation 60° anticlockwise about the origin.
	$\begin{pmatrix}d & 0\\ 0 & 1\end{pmatrix}$	B1	One-way stretch in the x-direction, scale factor d.
	$\mathbf{M}=\begin{pmatrix}d & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}=\begin{pmatrix}\frac{d}{2} & -\frac{d\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}$	M1 A1	Correct order.
		4	
4(b)	$d=\frac{1}{2}d^2$	M1	Uses value of $\det \mathbf{M}$.
	$d\neq 0 \Rightarrow d=2$	A1	AG
		2	

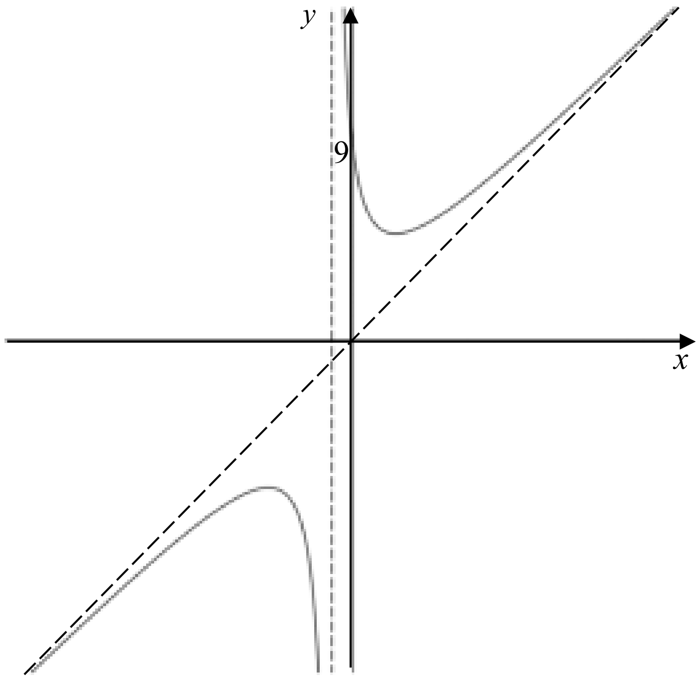
Question	Answer	Marks	Guidance
4(c)	$\mathbf{M}^{-1}=\frac{1}{2}\begin{pmatrix}\frac{1}{2} & \sqrt{3}\\ -\frac{\sqrt{3}}{2} & 1\end{pmatrix}$	B1 FT	Inverse of *their* **M**.
	$\mathbf{N}=\mathbf{M}^{-1}\begin{pmatrix}1 & 1\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}\frac{1}{2} & \sqrt{3}\\ -\frac{\sqrt{3}}{2} & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}$	M1	Multiplies on the left by the inverse of *their* **M**.
	$=\frac{1}{4}\begin{pmatrix}1+\sqrt{3} & 1+\sqrt{3}\\ 1-\sqrt{3} & 1-\sqrt{3}\end{pmatrix}$	A1	CAO
		3	
4(d)	$\begin{pmatrix}1 & 1\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x+y\\ \frac{1}{2}x+\frac{1}{2}y\end{pmatrix}$	B1	Transforms $\begin{pmatrix}x\\ y\end{pmatrix}$ to $\begin{pmatrix}X\\ Y\end{pmatrix}$.
	$\frac{1}{2}x+\frac{1}{2}mx=m(x+mx)$	M1 A1	Uses $y=mx$ and $Y=mX$.
	$1+m=2m+2m^2 \Rightarrow 2m^2+m-1=0$	A1	
	$y=\frac{1}{2}x$ and $y=-x$	A1	WWW
		5	

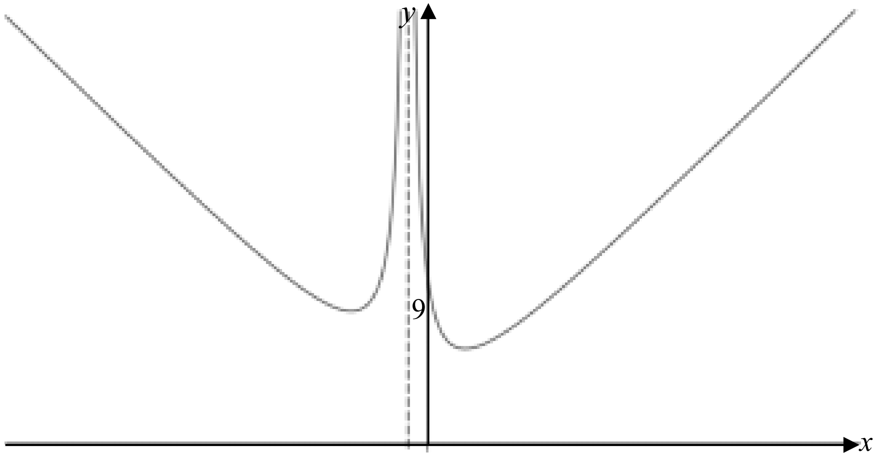
Question	Answer	Marks	Guidance
5(a)	$\theta = 0$ 	B1 B1	Initial line with correct position and shape of curve. SCB1 if correct shape with wrong starting position. Pole position must be clear.
	$a\cot\frac{1}{6}\pi = 2\sqrt{3} \Rightarrow a = 2$	B1	Substitutes $\theta = \frac{1}{6}\pi$. AG.
		3	
5(b)	$\frac{1}{2}\int_0^{\frac{1}{6}\pi} 4\cot^2\left(\frac{1}{3}\pi - \theta\right) \mathrm{d}\theta$	M1	Uses $\frac{1}{2}\int r^2\,\mathrm{d}\theta$ with correct limits.
	$2\int_0^{\frac{1}{6}\pi} \operatorname{cosec}^2\left(\frac{1}{3}\pi - \theta\right) - 1\,\mathrm{d}\theta = 2\left[\cot\left(\frac{1}{3}\pi - \theta\right) - \theta\right]_0^{\frac{1}{6}\pi}$	M1 A1	Uses $\cot^2\left(\frac{1}{3}\pi - \theta\right) = \operatorname{cosec}^2\left(\frac{1}{3}\pi - \theta\right) - 1$ and integrates.
	$2\left(\sqrt{3} - \frac{1}{6}\pi - \frac{1}{3}\sqrt{3}\right) = \frac{4}{3}\sqrt{3} - \frac{1}{3}\pi$	A1	OE, must be exact.
		4	
5(c)	$r = 2\dfrac{\cos\frac{\pi}{3}\cos\theta + \sin\frac{\pi}{3}\sin\theta}{\sin\frac{\pi}{3}\cos\theta - \cos\frac{\pi}{3}\sin\theta}$	M1	Uses the identities for $\cos(A-B)$ and $\sin(A-B)$. Or identity for $\tan(A-B)$.
	$\sqrt{x^2+y^2} = 2\dfrac{r\cos\theta + \sqrt{3}r\sin\theta}{\sqrt{3}r\cos\theta - r\sin\theta}$	M1	Applies $r = \sqrt{x^2+y^2}$, $x = r\cos\theta$ and $y = r\sin\theta$.
	$2\left(x + y\sqrt{3}\right) = \left(x\sqrt{3} - y\right)\sqrt{x^2+y^2}$	A1	AG.
		3	

Question	Answer	Marks	Guidance
6(a)	$\begin{pmatrix}0\\1\\t\end{pmatrix}-\begin{pmatrix}t\\1\\0\end{pmatrix}=t\begin{pmatrix}-1\\0\\1\end{pmatrix}$	B1	Finds vector from any point on l_1 to any point on l_2.
	$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\-2 & -1 & 0\\0 & -2 & 1\end{vmatrix}=\begin{pmatrix}-1\\2\\4\end{pmatrix}$	M1 A1	Finds common perpendicular.
	$\frac{t}{\sqrt{21}}\left\|\begin{pmatrix}-1\\0\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\4\end{pmatrix}\right\|=\sqrt{21}\Rightarrow t=\frac{21}{5}=4.2$	M1 A1	Uses formula for shortest distance.
		5	
6(b)	$\mathbf{r}=\frac{21}{5}\mathbf{i}+\mathbf{j}+\lambda(2\mathbf{i}+\mathbf{j})+\mu(-2\mathbf{j}+\mathbf{k})$	B1 FT	Using their value of t.
		1	
6(c)	$\begin{pmatrix}0\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}5\\-6\\7\end{pmatrix}=\sqrt{5}\sqrt{110}\cos\alpha$ leading to $\cos\alpha=\frac{19}{\sqrt{5}\sqrt{110}}$	M1 A1	Uses dot product of $-2\mathbf{j}+\mathbf{k}$ and normal.
	Acute angle between l_2 and Π_2 is $90-\alpha=54.1^\circ$	A1	
		3	

Question	Answer	Marks	Guidance
6(d)	$\begin{pmatrix}-1\\2\\4\end{pmatrix}\bullet\begin{pmatrix}5\\-6\\7\end{pmatrix}=\sqrt{21}\sqrt{110}\cos\alpha$ leading to $\cos\alpha=\dfrac{11}{\sqrt{21}\sqrt{110}}$	M1 A1 FT	Dot product using their normal to Π_1.
	76.8°	A1	
		3	

Question	Answer	Marks	Guidance
7(a)	$x=-1$	B1	States vertical asymptote.
	$y=\dfrac{x(x+1)+9}{x+1}=x+\dfrac{9}{x+1}$	M1	Finds oblique asymptote.
	$y=x$	A1	
		3	
7(b)	$\dfrac{dy}{dx}=1-9(x+1)^{-2}=0\Rightarrow(x+1)^2=9$	M1 A1	Differentiates and sets derivative equal to 0.
	$(2, 5)$	A1	
	$(-4, -7)$	A1	
		4	

Question	Answer	Marks	Guidance
7(c)		B1	Axes labelled and correct asymptotes drawn.
		B1	Upper branch with $(0, 9)$ stated or shown on diagram.
		B1	Lower branch correct and good approach to asymptotes throughout, no extra branches.
		3	

Question	Answer	Marks	Guidance
7(d)		B1 FT	FT from sketch in **(c)** with asymptotes shown.
	$x^2+x+9=\frac{13}{2}(x+1)$ or $x^2+x+9=-\frac{13}{2}(x+1)$ $x^2-\frac{11}{2}x+\frac{5}{2}=0$ or $x^2+\frac{15}{2}x+\frac{31}{2}=0$	M1 M1	Finds critical points, award M1 for each case. May state that $x^2+x+9=-\frac{13}{2}(x+1)$ has no real solutions since $7>\frac{13}{2}$.
	$x=\frac{1}{2}, 5$	A1	
	$x<\frac{1}{2}$ and $x>5$.	A1	
		5	

Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

5202311540

FURTHER MATHEMATICS **9231/13**

Paper 1 Further Pure Mathematics 1 **May/June 2021**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **20** pages. Any blank pages are indicated.

DC (PQ) 199004/2

1 **(a)** Show that

$$\tan(r+1) - \tan r = \frac{\sin 1}{\cos(r+1)\cos r}.$$ [2]

Let $u_r = \dfrac{1}{\cos(r+1)\cos r}$.

(b) Use the method of differences to find $\sum_{r=1}^{n} u_r$. [3]

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(c) Explain why the infinite series $u_1 + u_2 + u_3 + \dots$ does not converge. [1]

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2 The cubic equation $2x^3 - 4x^2 + 3 = 0$ has roots α, β, γ. Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(a) State the value of S_1 and find the value of S_2. [3]

(b) **(i)** Express S_{n+3} in terms of S_{n+2} and S_n. [1]

(ii) Hence, or otherwise, find the value of S_4. [2]

(c) Use the substitution $y = S_1 - x$, where S_1 is the numerical value found in part **(a)**, to find and simplify an equation whose roots are $\alpha+\beta$, $\beta+\gamma$, $\gamma+\alpha$. [3]

(d) Find the value of $\dfrac{1}{\alpha+\beta}+\dfrac{1}{\beta+\gamma}+\dfrac{1}{\gamma+\alpha}$. [2]

3 **(a)** Prove by mathematical induction that, for all positive integers n,

$$\sum_{r=1}^{n}\left(5r^4 + r^2\right) = \tfrac{1}{2}n^2(n+1)^2(2n+1).$$ [6]

(b) Use the result given in part **(a)** together with the List of formulae (MF19) to find $\sum_{r=1}^{n} r^4$ in terms of n, fully factorising your answer. [3]

4 The matrices **A**, **B** and **C** are given by

$$\mathbf{A} = \begin{pmatrix} 2 & k & k \\ 5 & -1 & 3 \\ 1 & 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \quad \mathbf{C} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 2 & 0 \end{pmatrix},$$

where k is a real constant.

(a) Find **CAB**. [3]

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(b) Given that **A** is singular, find the value of k. [3]

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(c) Using the value of k from part **(b)**, find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **CAB**. [5]

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5 The curve C has polar equation $r = \frac{1}{\pi - \theta} - \frac{1}{\pi}$, where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) Sketch C. [3]

(b) Show that the area of the region bounded by the half-line $\theta = \frac{1}{2}\pi$ and C is $\frac{3 - 4\ln 2}{4\pi}$. [6]

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6 The lines l_1 and l_2 have equations $\mathbf{r} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k} + s(2\mathbf{i} - 3\mathbf{j})$ and $\mathbf{r} = 3\mathbf{i} - 2\mathbf{k} + t(3\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ respectively.

The plane Π_1 contains l_1 and the point P with position vector $-2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$.

(a) Find an equation of Π_1, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$. [2]

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The plane Π_2 contains l_2 and is parallel to l_1.

(b) Find an equation of Π_2, giving your answer in the form $ax + by + cz = d$. [4]

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(c) Find the acute angle between Π_1 and Π_2. [5]

(d) The point Q is such that $\overrightarrow{OQ} = -5\overrightarrow{OP}$.

Find the position vector of the foot of the perpendicular from the point Q to Π_2. [4]

7 The curve C has equation $y = \dfrac{x^2 - x - 3}{1 + x - x^2}$.

(a) Find the equations of the asymptotes of C. [2]

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(b) Find the coordinates of any stationary points on C. [3]

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(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

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(d) Sketch the curve with equation $y = \left|\dfrac{x^2 - x - 3}{1 + x - x^2}\right|$ and find in exact form the set of values of x for which $\left|\dfrac{x^2 - x - 3}{1 + x - x^2}\right| < 3$. [6]

Additional Page

If you use the following lined page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.

Cambridge Assessment International Education

Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/13**

Paper 1 Further Pure Mathematics 1 **May/June 2021**

MARK SCHEME

Maximum Mark: 75

Published

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Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

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This document consists of **16** printed pages.

Question	Answer	Marks	Guidance
1(a)	$\frac{\sin(r+1)}{\cos(r+1)}-\frac{\sin r}{\cos r}=\frac{\sin(r+1)\cos r-\cos(r+1)\sin r}{\cos(r+1)\cos r}=\frac{\sin(r+1-r)}{\cos(r+1)\cos r}$ OR $\frac{\sin(r+1)}{\cos(r+1)}-\frac{\sin r}{\cos r}=\frac{\sin r\cos 1+\cos r\sin 1}{\cos(r+1)}-\frac{\sin r}{\cos r}$ $=\frac{\sin r\cos r\cos 1+\cos^2 r\sin 1-\sin r\cos r\cos 1+\sin^2 r\sin 1}{\cos(r+1)\cos r}$	M1	Applies all relevant correct addition formulae from MF19 for the route chosen and combines into a single fraction.
	$=\frac{\sin 1}{\cos(r+1)\cos r}$ $\frac{\tan r+\tan 1}{1-\tan r\tan 1}-\tan r=\frac{\tan r+\tan 1-\tan r+\tan^2 r\tan 1}{1-\tan r\tan 1}$ $=\frac{\tan 1\sec^2 r}{1-\tan r\tan 1}=\frac{\tan 1}{\cos r(\cos r-\sin r\tan 1)}=\frac{\sin 1}{\cos r(\cos r\cos 1-\sin r\sin 1)}$ $=\frac{\sin 1}{\cos r\cos(r+1)}$	A1	SC B2 for this route completely correct to AG.
		2	
1(b)	$[\sin 1]\sum_{r=1}^{n}u_r=\tan 2-\tan 1+\tan 3-\tan 2+\ldots+\tan(n+1)-\tan n$	M1 A1	Shows enough terms for cancellation to be clear. There must be three complete terms including first and last. Cancelling may be implied.
	$\sum_{r=1}^{n}u_r=\frac{\tan(n+1)-\tan 1}{\sin 1}$	A1	OE and ISW.
		3	

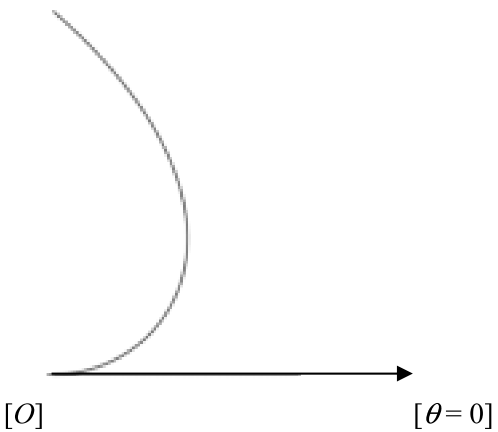
Question	Answer	Marks	Guidance
1(c)	$\tan(n+1)$ oscillates as $n \to \infty$ so $u_1 + u_2 + u_3 + \dots$ does not converge.	B1	States 'oscillates' or refers to diverging values of $\tan(n+1)$, or states that $\tan(n+1)$ does not tend to a limit.
		1	
2(a)	$S_1 = 2$	B1	
	$S_2 = S_1{}^2 - 2(0)$	M1	Uses formula for sum of squares.
	$= 4$	A1	Correct answer implies M1A1.
		3	
2(b)(i)	$S_{n+3} = 2S_{n+2} - \frac{3}{2}S_n$	B1	CAO or as a single fraction.
		1	
2(b)(ii)	$S_4 = 2S_3 - \frac{3}{2}S_1 = 2(2S_2 - \frac{3}{2}S_0) - \frac{3}{2}S_1$	M1	Uses their recursive formula from part **(i)** to find S_4 $\left[S_3 = \frac{7}{2}\right]$.
	$= 4$	A1	
		2	
2(c)	$x = 2 - y$	B1	SOI
	$2(2-y)^3 - 4(2-y)^2 + 3 = 0$	M1	Makes *their* substitution.
	$2y^3 - 8y^2 + 8y - 3 = 0$	A1	OE but must be an equation.
		3	

Question	Answer	Marks	Guidance
2(d)	$\dfrac{\frac{8}{2}}{\frac{-3}{2}}$ OR Or use $2S_2 - 8S_1 + 8S_0 - 3S_{-1} = 0$ with substitution of their values	M1	Uses $\dfrac{1}{\alpha'} + \dfrac{1}{\beta'} + \dfrac{1}{\gamma'} = \dfrac{\alpha'\beta' + \beta'\gamma' + \gamma'\alpha'}{\alpha'\beta'\gamma'}$.
	$= \dfrac{8}{3}$	A1 FT	FT from **2(c).**
		2	

Question	Answer	Marks	Guidance
3(a)	$5\times1^4+1^2=\frac{1}{2}(2)^2(2+1)[=6]$ so H_1 is true.	B1	Checks base case.
	Assume that $\sum_{r=1}^{k}\left[\left(5r^4+r^2\right)\right]=\frac{1}{2}k^2(k+1)^2(2k+1)$	B1	States inductive hypothesis [for some *k*] including the algebraic form. If says for **ALL** *k*, then B0.
	$\sum_{r=1}^{k+1}\left[\left(5r^4+r^2\right)\right]=\frac{1}{2}k^2(k+1)^2(2k+1)+5(k+1)^4+(k+1)^2$	M1	Considers sum to $k+1$.
	$\frac{1}{2}(k+1)^2\left(2k^3+k^2+10(k+1)^2+2\right)$	M1	Take out the factor of $(k+1)^2$ OR expands the summation expression and the target expression for $k+1$ and collects like terms for both.
	$\frac{1}{2}(k+1)^2(2k^3+11k^2+20k+12)=\frac{1}{2}(k+1)^2(k+2)^2(2k+3)$	A1	Factorises or having expanded, checks explicitly. At least one intermediate step seen following the award of M1 before reaching the answer.
	So H_{k+1} is true. By induction, H_n is true for all positive integers *n*.	A1	States conclusion. Implication must be clearly expressed.
		6	
3(b)	$5\sum_{r=1}^{n}r^4+\frac{1}{6}n(n+1)(2n+1)=\frac{1}{2}n^2(n+1)^2(2n+1)$	M1	Uses correct formula for $\sum r^2$.
	$[5]\sum_{r=1}^{n}r^4=\frac{1}{6}n(n+1)(2n+1)(3n(n+1)-1)$	M1	Makes $\sum r^4$ the subject and takes out all linear factors and the remaining term is of correct form.
	$\sum_{r=1}^{n}r^4=\frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$	A1	CAO
		3	

Question	Answer	Marks	Guidance
4(a)	$\begin{pmatrix}0&1&1\\-1&2&0\end{pmatrix}\begin{pmatrix}2&k&k\\5&-1&3\\1&0&1\end{pmatrix}\begin{pmatrix}1&0\\0&1\\1&0\end{pmatrix}=\begin{pmatrix}0&1&1\\-1&2&0\end{pmatrix}\begin{pmatrix}2+k&k\\8&-1\\2&0\end{pmatrix}$ Or $\begin{pmatrix}0&1&1\\-1&2&0\end{pmatrix}\begin{pmatrix}2&k&k\\5&-1&3\\1&0&1\end{pmatrix}\begin{pmatrix}1&0\\0&1\\1&0\end{pmatrix}=\begin{pmatrix}6&-1&4\\8&-k-2&-k+6\end{pmatrix}\begin{pmatrix}1&0\\0&1\\1&0\end{pmatrix}$	M1 A1	Multiplies two matrices correctly.
	$\begin{pmatrix}10&-1\\-k+14&-k-2\end{pmatrix}$	A1	
		3	
4(b)	$2\begin{vmatrix}-1&3\\0&1\end{vmatrix}-k\begin{vmatrix}5&3\\1&1\end{vmatrix}+k\begin{vmatrix}5&-1\\1&0\end{vmatrix}=0$ leading to $-2-2k+k=0$	M1 A1	Sets determinant equal to zero and forms linear equation.
	$k=-2$	A1	
		3	

Question	Answer	Marks	Guidance
4(c)	$\begin{pmatrix}10 & -1\\16 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}10x-y\\16x\end{pmatrix}$	M1	Transforms $\begin{pmatrix}x\\y\end{pmatrix}$. Allow $q\begin{pmatrix}1\\m\end{pmatrix}$ where q is x, t or a nonzero number.
	$10x-mx=X$ and $16x=mX$	M1 A1	Uses $y=mx$ and $Y=mX$. Expect $16x=m(10x-mx)$.
	$16=10m-m^2$ $[m^2-10m+16=0]$	A1	OE
	$y=2x$ and $y=8x$	A1	
		5	

Question	Answer	Marks	Guidance
5(a)		B1	Approximately correct curve passing through the pole, O, with correct domain.
		B1	r strictly increasing over the domain 0 to $\frac{\pi}{2}$.
		B1	Correct form at O. Approx tangential to initial line at O.
		3	

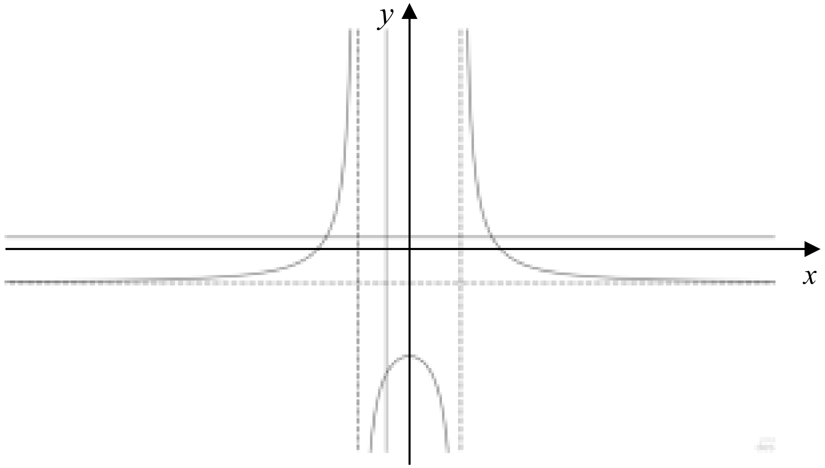
Question	Answer	Marks	Guidance
5(b)	$\frac{1}{2}\int_0^{\frac{1}{2}\pi}\left(\frac{1}{\pi-\theta}-\frac{1}{\pi}\right)^2 \, d\theta$	M1	Uses $\frac{1}{2}\int r^2 \, d\theta$ with correct limits.
	$\frac{1}{2}\int_0^{\frac{1}{2}\pi}\frac{1}{(\pi-\theta)^2}-\frac{2}{\pi(\pi-\theta)}+\frac{1}{\pi^2}\,d\theta$	M1	Expands.
	$\frac{1}{2}\left[\frac{1}{\pi-\theta}+\frac{2}{\pi}\ln(\pi-\theta)+\frac{\theta}{\pi^2}\right]_0^{\frac{1}{2}\pi}$	M1 A1	Integrates all terms to obtain correct form.
	$\frac{1}{2}\left(\frac{2}{\pi}+\frac{2}{\pi}\ln\frac{\pi}{2}+\frac{1}{2\pi}-\left(\frac{1}{\pi}+\frac{2}{\pi}\ln\pi\right)\right)=\frac{1}{2}\left(\frac{3}{2\pi}+\frac{2}{\pi}\ln\frac{1}{2}\right)$	M1	Substitute limits into an expression of the correct form and simplify the log terms.
	$=\frac{3-4\ln 2}{4\pi}$	A1	AG
		6	

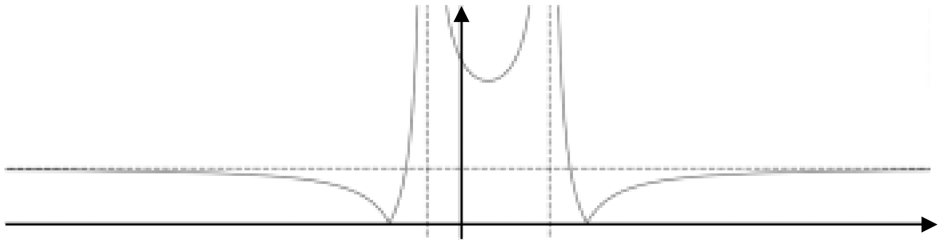
Question	Answer	Marks	Guidance
6(a)	$-(-2\mathbf{i}-2\mathbf{j}+4\mathbf{k})+(-\mathbf{i}-2\mathbf{j}+\mathbf{k})=\mathbf{i}-3\mathbf{k}$	B1	OE. Finds direction vector from P to a point of l_1.
	$\mathbf{r}=-2\mathbf{i}-2\mathbf{j}+4\mathbf{k}+\lambda(2\mathbf{i}-3\mathbf{j})+\mu(\mathbf{i}-3\mathbf{k})$ $\boldsymbol{r}=-\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k}+\lambda(2i-3j)+\mu(\boldsymbol{i}-3\boldsymbol{k})$	B1 FT	OE. FT *their* $\mathbf{i}-3\mathbf{k}$
		2	

Question	Answer	Marks	Guidance
6(b)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 0 \\ 3 & -1 & 3 \end{vmatrix} = \begin{pmatrix} -9 \\ -6 \\ 7 \end{pmatrix}$	M1 A1	OE. Finds vector perpendicular to Π_2.
	$-9(3) - 6(0) + 7(-2) = -41$	M1	Uses point on Π_2.
	$9x + 6y - 7z = 41$	A1	
		4	
6(c)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 0 \\ 1 & 0 & -3 \end{vmatrix} = \begin{pmatrix} 9 \\ 6 \\ 3 \end{pmatrix} \sim \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$	M1 A1	Finds vector perpendicular to Π_1.
	$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 6 \\ -7 \end{pmatrix} = \sqrt{14}\sqrt{166}\cos\alpha$ leading to $\cos\alpha = \frac{32}{\sqrt{14}\sqrt{166}}$	DM1 A1 FT	Dot product using their normal vectors.
	48.4°	A1	0.845 rad
		5	

Question	Answer	Marks	Guidance
6(d)	$\overrightarrow{OF} = \overrightarrow{OQ} + \overrightarrow{QF} = -5\begin{pmatrix}-2\\-2\\4\end{pmatrix} + t\begin{pmatrix}9\\6\\-7\end{pmatrix} = \begin{pmatrix}10+9t\\10+6t\\-20-7t\end{pmatrix}$	M1 A1 FT	Scales $\overrightarrow{OP}$ by a factor of ±5 and uses multiple of their normal to Π_2.
	$9(10+9t)+6(10+6t)-7(-20-7t)=41$ leading to $290+166t=41$	DM1	Substitutes into the equation of Π_2.
	$t=-\frac{3}{2}$ leading to $\overrightarrow{OF} = \begin{pmatrix}-\frac{7}{2}\\1\\-\frac{19}{2}\end{pmatrix}$	A1	
		4	

Question	Answer	Marks	Guidance
7(a)	$x=\frac{1-\sqrt{5}}{2}$, $x=\frac{1+\sqrt{5}}{2}$	B1	Vertical asymptotes. Must be exact.
	$y=-1$	B1	Horizontal asymptote.
		2	
7(b)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(1+x-x^2)(2x-1)-(x^2-x-3)(1-2x)}{(1+x-x^2)^2}$	M1	Finds $\frac{\mathrm{d}y}{\mathrm{d}x}$.
	$(2x-1)(-2)=0$	M1	Sets their $\frac{dy}{dx}$ equal to 0 and forms equation.
	$\left(\frac{1}{2},-\frac{13}{5}\right)$	A1	WWW
		3	

Question	Answer	Marks	Guidance
7(c)		B1	Both axes labelled and correct asymptotes shown.
		B1	Correct shape and position, with all asymptotic behaviour clear.
	$\left(\frac{1}{2}+\frac{1}{2}\sqrt{13},0\right)$, $\left(\frac{1}{2}-\frac{1}{2}\sqrt{13},0\right)$, $(0,-3)$	B1	States exact coordinates of intersections with axes.
		3	

Question	Answer	Marks	Guidance
7(d)		B1 FT	FT from sketch in **(c).**
		B1	Correct shape as x tends to infinity and intersections with x axis.
	$\frac{x^2-x-3}{1+x-x^2}=3$ or $\frac{x^2-x-3}{1+x-x^2}=-3$ $4x^2-4x-6=0$ or $-2x^2+2x=0$	M2	Finds critical points, award M1 for each case.
	$x=\frac{1}{2}+\frac{1}{2}\sqrt{7}$, $x=\frac{1}{2}-\frac{1}{2}\sqrt{7}$, $x=0$, $x=1$	A1	Must be exact.
	$x<\frac{1}{2}-\frac{1}{2}\sqrt{7}$, $0<x<1$, $x>\frac{1}{2}+\frac{1}{2}\sqrt{7}$	A1 FT	Must be three distinct regions and strict inequalities.
		6	