Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

6916036546

FURTHER MATHEMATICS **9231/11**

Paper 1 Further Pure Mathematics 1 **October/November 2020**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **20** pages. Blank pages are indicated.

DC (RW) 187465/2

1 The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, where a and b are positive constants.

(a) The matrix **M** represents a sequence of two geometrical transformations.

State the type of each transformation, and make clear the order in which they are applied. [2]

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The unit square in the x-y plane is transformed by **M** onto parallelogram $OPQR$.

(b) Find, in terms of a and b, the matrix which transforms parallelogram $OPQR$ onto the unit square. [2]

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It is given that the area of $OPQR$ is $2\,\text{cm}^2$ and that the line $x + 3y = 0$ is invariant under the transformation represented by $\mathbf{M}$.

(c) Find the values of a and b. [5]

2 **(a)** Use standard results from the List of Formulae (MF19) to show that

$$\sum_{r=1}^{n} (7r+1)(7r+8) = an^3 + bn^2 + cn,$$

where a, b and c are constants to be determined. [3]

(b) Use the method of differences to find $\sum_{r=1}^{n} \frac{1}{(7r+1)(7r+8)}$ in terms of n. [4]

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(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{(7r+1)(7r+8)}$. [1]

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3 The cubic equation $x^3 + cx + 1 = 0$, where c is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are α^3, β^3, γ^3. [3]

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(b) Show that $\alpha^6 + \beta^6 + \gamma^6 = 3 - 2c^3$. [3]

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(c) Find the real value of c for which the matrix $\begin{pmatrix} 1 & \alpha^3 & \beta^3 \\ \alpha^3 & 1 & \gamma^3 \\ \beta^3 & \gamma^3 & 1 \end{pmatrix}$ is singular. [5]

4 The points A, B, C have position vectors

$$-\mathbf{i}+\mathbf{j}+2\mathbf{k}, \quad -2\mathbf{i}-\mathbf{j}, \quad 2\mathbf{i}+2\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

(b) Find the perpendicular distance from O to the plane ABC. [2]

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(c) Find the acute angle between the planes OAB and ABC. [4]

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5 Prove by mathematical induction that, for every positive integer n,

$$\frac{\mathrm{d}^{2n-1}}{\mathrm{d}x^{2n-1}}(x\sin x) = (-1)^{n-1}\big(x\cos x + (2n-1)\sin x\big).$$ [7]

6 The curve C has equation $y = \dfrac{x^2 + x - 1}{x - 1}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Show that there is no point on C for which $1 < y < 5$. [4]

(c) Find the coordinates of the intersections of C with the axes, and sketch C. [3]

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(d) Sketch the curve with equation $y = \left|\dfrac{x^2 + x - 1}{x - 1}\right|$. [2]

7 **(a)** Show that the curve with Cartesian equation

$$(x^2+y^2)^{\frac{5}{2}} = 4xy(x^2-y^2)$$

has polar equation $r = \sin 4\theta$. [4]

The curve C has polar equation $r = \sin 4\theta$, for $0 \leqslant \theta \leqslant \frac{1}{4}\pi$.

(b) Sketch C and state the equation of the line of symmetry. [3]

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(c) Find the exact value of the area of the region enclosed by C. [4]

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(d) Using the identity $\sin 4\theta \equiv 4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta$, find the maximum distance of C from the line $\theta = \frac{1}{2}\pi$. Give your answer correct to 2 decimal places. [6]

Additional Page

If you use the following lined page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.

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FURTHER MATHEMATICS **9231/11**

Paper 1 Further Pure Mathematics 1 **October/November 2020**

MARK SCHEME

Maximum Mark: 75

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

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This document consists of **15** printed pages.

Question	Answer	Marks	Guidance
1(a)	One-way stretch followed by a shear.	**B2**	Both named correctly. Award B1 if given in the wrong order.
		2	
1(b)	$\mathbf{M}^{-1} = \frac{1}{a}\begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix}$	**M1 A1**	
		2	
1(c)	$a = 2$	**B1**	
	$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ y \end{pmatrix}$	**B1**	Transforms $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} X \\ Y \end{pmatrix}$.
	$= \begin{pmatrix} ax - \frac{1}{3}bx \\ -\frac{1}{3}x \end{pmatrix}$	**M1**	Uses $x + 3y = 0$.
	$x = ax - \frac{1}{3}bx \Rightarrow 1 = a - \frac{1}{3}b$	**M1**	Uses that line is invariant (or $X + 3Y = 0$).
	$b = 3$	**A1**	
		5	

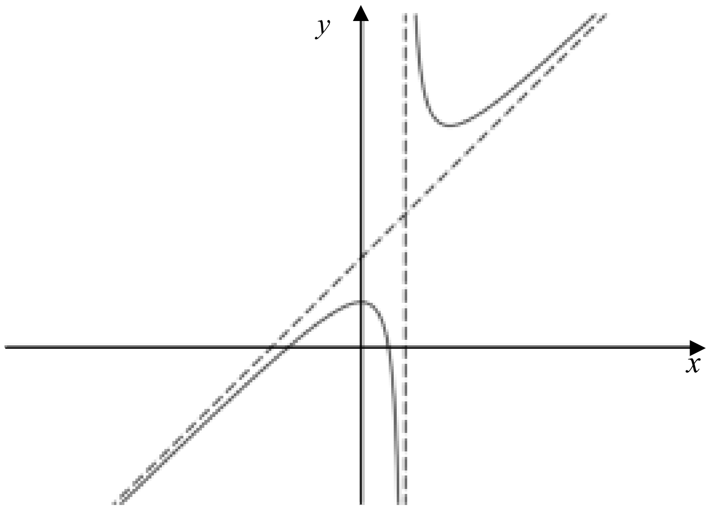
Question	Answer	Marks	Guidance
2(a)	$\sum_{r=1}^{n}(7r+1)(7r+8)=\sum_{r=1}^{n}49r^2+63r+8$	M1	Expands.
	$=49\left(\frac{1}{6}n(n+1)(2n+1)\right)+63\left(\frac{1}{2}n(n+1)\right)+8n$	M1	Substitutes formulae for $\sum r^2$ and $\sum r$.
	$=\frac{49}{3}n^3+56n^2+\frac{143}{3}n$	A1	
		3	
2(b)	$\frac{1}{(7r+1)(7r+8)}=\frac{1}{7}\left(\frac{1}{7r+1}-\frac{1}{7r+8}\right)$	M1 A1	Finds partial fractions.
	$\sum_{r=1}^{n}\frac{1}{(7r+1)(7r+8)}=\frac{1}{7}\left(\frac{1}{8}-\frac{1}{15}+\frac{1}{15}-\frac{1}{22}+\ldots+\frac{1}{7n+1}-\frac{1}{7n+8}\right)$	M1	Writes at least three correct terms, including first and last.
	$=\frac{1}{7}\left(\frac{1}{8}-\frac{1}{7n+8}\right)$	A1	
		4	
2(c)	$\frac{1}{56}$	B1 FT	
		1	

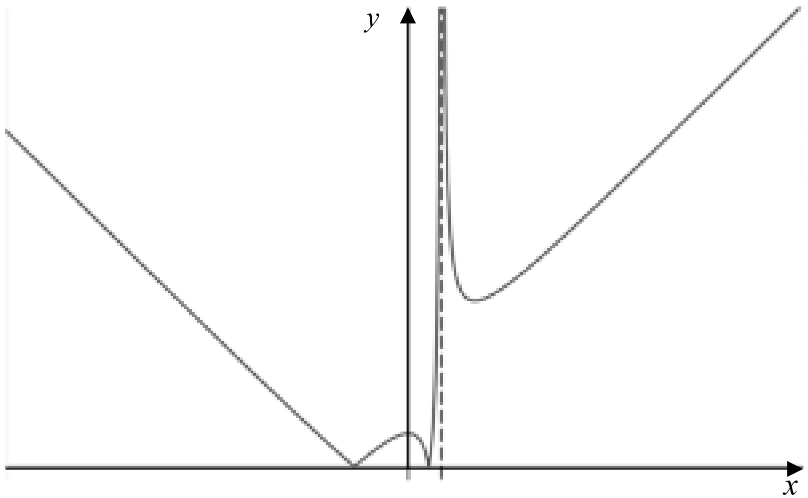
Question	Answer	Marks	Guidance
3(a)	$y = x^3 \Rightarrow x = y^{\frac{1}{3}}$	B1	Substitutes.
	$y + cy^{\frac{1}{3}} + 1 = 0 \Rightarrow -c^3 y = (y+1)^3 = y^3 + 3y^2 + 3y + 1$	M1	Correct attempt to eliminate cube root.
	$y^3 + 3y^2 + (3 + c^3)y + 1 = 0$	A1	
		3	
3(b)	$\alpha^3 + \beta^3 + \gamma^3 = -3 \quad \alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3 = 3 + c^3$	B1 FT	Using *their* answer to **(a)**.
	$\alpha^6 + \beta^6 + \gamma^6 = (-3)^2 - 2(3 + c^3)$	M1	$\alpha^6 + \beta^6 + \gamma^6 = (\alpha^3 + \beta^3 + \gamma^3)^2 - 2(\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3)$
	$= 3 - 2c^3$	A1	AG
		3	
3(c)	$\alpha^3\beta^3\gamma^3 = -1$	B1	If using *their* answer to **(a)** FT
	$\begin{vmatrix} 1 & \alpha^3 & \beta^3 \\ \alpha^3 & 1 & \gamma^3 \\ \beta^3 & \gamma^3 & 1 \end{vmatrix} = 1 - (\alpha^6 + \beta^6 + \gamma^6) + 2\alpha^3\beta^3\gamma^3 = 2c^3 - 4$	M1 A1	Evaluates determinant.
	$2c^3 - 4 = 0$	M1	Sets determinant equal to zero.
	$c = \sqrt[3]{2}$	A1	
		5	

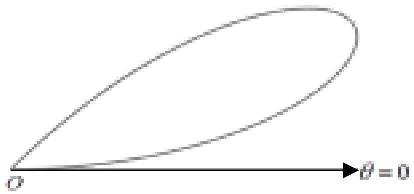
Question	Answer	Marks	Guidance
4(a)	$\overrightarrow{AB} = -\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ $\overrightarrow{AC} = 3\mathbf{i} - \mathbf{j}$	B1	Finds direction vectors of two lines in the plane. $\overrightarrow{BC} = 4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 3 & -1 & 0 \end{vmatrix} = \begin{pmatrix} -2 \\ -6 \\ 7 \end{pmatrix}$	M1 A1	Finds normal to the plane *ABC*.
	$-2(-1) - 6(1) + 7(2) = 10 \Rightarrow -2x - 6y + 7z = 10$	M1 A1	Substitutes point.
	Alternative method for question 4(a)		
	Setting up 3 equations using points given.	M1	
	$-2x - 6y + 7z = 10$	A1 A1 A1 A1	OE
		5	
4(b)	$\dfrac{10}{\sqrt{2^2 + 6^2 + 7^2}} = \dfrac{10}{\sqrt{89}}$ OE	M1 A1	Divides by magnitude of normal vector. 1.06…
		2	

Question	Answer	Marks	Guidance
4(c)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 2 \\ -2 & -1 & 0 \end{vmatrix} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}$	M1 A1	Finds normal to the plane *OAB*.
	$\begin{pmatrix} -2 \\ -6 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} = \sqrt{89}\sqrt{29}\cos\theta$	M1	Uses dot product correctly.
	36.2°	A1	
		4	

Question	Answer	Marks	Guidance
5	$\frac{\mathrm{d}}{\mathrm{d}x}(x\sin x) = x\cos x + \sin x = (-1)^0(x\cos x + (2(1)-1)\sin x)$	B1	Checks base case using product rule.
	Assume true for $n = k$, so $\frac{\mathrm{d}^{2k-1}}{\mathrm{d}x^{2k-1}}(x\sin x) = (-1)^{k-1}(x\cos x + (2k-1)\sin x)$	B1	States inductive hypothesis.
	Then $\frac{\mathrm{d}^{2k}}{\mathrm{d}x^{2k}}(x\sin x) = (-1)^{k-1}(-x\sin x + 2k\cos x)$	M1 A1	Differentiates once. Must have correct LHS for A1.
	$\frac{\mathrm{d}^{2k+1}}{\mathrm{d}x^{2k+1}}(x\sin x) = (-1)^{k-1}(-x\cos x - \sin x - 2k\sin x)$ $= (-1)^k(x\cos x + (2k+1)\sin x)$	M1 A1	Differentiates again.
	So, it is also true for $n = k+1$. Hence, by induction, true for all positive integers.	A1	States conclusion.
		7	

Question	Answer	Marks	Guidance
6(a)	$x=1$	**B1**	States vertical asymptote.
	$x^2+x-1=(x-1)(x+2)+1 \Rightarrow y=x+2$	**M1 A1**	Finds oblique asymptote.
		3	
6(b)	$yx-y=x^2+x-1 \Rightarrow x^2+(1-y)x+y-1=0$	**M1 A1**	Forms quadratic in x.
	$(1-y)^2-4(y-1)<0 \Rightarrow y^2-6y+5<0$	**M1**	Uses that discriminant is negative if there are no values of x.
	$1<y<5$	**A1**	AG
		4	
6(c)		**B1**	Axes and asymptotes.
		B1	Branches correct.
	$(0,1)$, $\left(-\frac{1}{2}+\frac{1}{2}\sqrt{5},0\right)$, $\left(-\frac{1}{2}-\frac{1}{2}\sqrt{5},0\right)$	**B1**	States coordinates of intersections with axes, can be labelled on graph. Accept $(0.618,0)$ and $(-1.62,0)$.
		3	

Question	Answer	Marks	Guidance
6(d)		**B1 FT**	FT from sketch in **(c)** with both branches.
		B1	Correct shape at extremities.
		2	

Question	Answer	Marks	Guidance
7(a)	$r^5 = 4xy\left(x^2 - y^2\right)$	B1	Uses $r^2 = x^2 + y^2$.
	$r^5 = 4r^4 \cos\theta \sin\theta\left(\cos^2\theta - \sin^2\theta\right)$	B1	Uses $x = r\cos\theta$ and $y = r\sin\theta$.
	$r = 2\sin 2\theta \cos 2\theta$	M1	Applies at least one double angle formula.
	$r = \sin 4\theta$	A1	Applies both double angle formulae, AG.
		4	
7(b)		B1	Initial line drawn and one loop in the first quadrant.
		B1	Correct shape at extremities.
	$\theta = \frac{1}{8}\pi$	B1	States the equation of the line of symmetry.
		3	
7(c)	$\frac{1}{2}\int_0^{\frac{1}{4}\pi} \sin^2 4\theta \, d\theta$	M1	Uses $\frac{1}{2}\int r^2 \, d\theta$ with correct limits.
	$= \frac{1}{4}\int_0^{\frac{1}{4}\pi} 1 - \cos 8\theta \, d\theta = \frac{1}{4}\left[\theta - \frac{1}{8}\sin 8\theta\right]_0^{\frac{1}{4}\pi}$	M1 A1	Applies double angle formula and integrates.
	$= \frac{1}{16}\pi$	A1	
		4	

Question	Answer	Marks	Guidance
7(d)	$x = 4\left(\sin\theta\cos^4\theta - \sin^3\theta\cos^2\theta\right)$	B1	Uses $x = r\cos\theta$.
	$-4\sin^2\theta\cos^3\theta + \cos^5\theta + 2\sin^4\theta\cos\theta - 3\sin^2\theta\cos^3\theta = 0$	M1	Differentiates and sets equal to 0.
	$\cos\theta\left(2\sin^4\theta - 7\sin^2\theta\cos^2\theta + \cos^4\theta\right) = 0$	A1	
	$\cos\theta = 0$ or $2\tan^4\theta - 7\tan^2\theta + 1 = 0$	M1	Forms quadratic in $\tan^2\theta$. Must see consideration of $\cos\theta = 0$.
	$\tan^2\theta = \frac{1}{4}\left(7 \pm \sqrt{41}\right) \Rightarrow \theta = \pm 0.369, \pm 1.071$	B1	Allow use of decimals.
	$\theta = 0.369 \Rightarrow x = 0.93$ (or $\theta = -0.369 \Rightarrow \lvert x \rvert = 0.93$)	A1	Substituting $\theta = \pm 0.369$ gives maximum value of $\lvert x \rvert$.
		6	

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9117044394

FURTHER MATHEMATICS **9231/12**

Paper 1 Further Pure Mathematics 1 **October/November 2020**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

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- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Blank pages are indicated.

DC (RW) 187466/1

1 The cubic equation $x^3 + bx^2 + cx + d = 0$, where b, c and d are constants, has roots α, β, γ. It is given that $\alpha\beta\gamma = -1$.

(a) State the value of d. [1]

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(b) Find a cubic equation, with coefficients in terms of b and c, whose roots are $\alpha + 1$, $\beta + 1$, $\gamma + 1$. [3]

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(c) Given also that $\gamma + 1 = -\alpha - 1$, deduce that $(c - 2b + 3)(b - 3) = b - c$. [4]

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2 Prove by mathematical induction that $7^{2n} - 1$ is divisible by 12 for every positive integer n. [5]

3 **(a)** By simplifying $\left(x^n - \sqrt{x^{2n}+1}\right)\left(x^n + \sqrt{x^{2n}+1}\right)$, show that $\dfrac{1}{x^n - \sqrt{x^{2n}+1}} = -x^n - \sqrt{x^{2n}+1}$. [1]

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Let $u_n = x^{n+1} + \sqrt{x^{2n+2}+1} + \dfrac{1}{x^n - \sqrt{x^{2n}+1}}$.

(b) Use the method of differences to find $\sum_{n=1}^{N} u_n$ in terms of N and x. [3]

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(c) Deduce the set of values of x for which the infinite series

$$u_1 + u_2 + u_3 + \dots$$

is convergent and give the sum to infinity when this exists. [3]

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4 The matrices **A** and **B** are given by

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}.$$

(a) Give full details of the geometrical transformation in the x-y plane represented by **A**. [1]

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(b) Give full details of the geometrical transformation in the x-y plane represented by **B**. [2]

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The triangle DEF in the x-y plane is transformed by **AB** onto triangle PQR.

(c) Show that the triangles DEF and PQR have the same area. [3]

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(d) Find the matrix which transforms triangle PQR onto triangle DEF. [2]

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(e) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **AB**. [5]

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5 The curve C has polar equation $r = \ln(1 + \pi - \theta)$, for $0 \leqslant \theta \leqslant \pi$.

(a) Sketch C and state the polar coordinates of the point of C furthest from the pole. [3]

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(b) Using the substitution $u = 1 + \pi - \theta$, or otherwise, show that the area of the region enclosed by C and the initial line is

$$\tfrac{1}{2}(1+\pi)\ln(1+\pi)\big(\ln(1+\pi) - 2\big) + \pi.$$ [6]

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(c) Show that, at the point of C furthest from the initial line,

$$(1+\pi-\theta)\ln(1+\pi-\theta)-\tan\theta=0$$

and verify that this equation has a root between 1.2 and 1.3. [5]

6 Let a be a positive constant.

(a) The curve C_1 has equation $y = \dfrac{x-a}{x-2a}$. [2]

Sketch C_1.

The curve C_2 has equation $y = \left(\dfrac{x-a}{x-2a}\right)^2$. The curve C_3 has equation $y = \left|\dfrac{x-a}{x-2a}\right|$.

(b) **(i)** Find the coordinates of any stationary points of C_2. [3]

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(ii) Find also the coordinates of any points of intersection of C_2 and C_3. [3]

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(c) Sketch C_2 and C_3 on a single diagram, clearly identifying each curve. Hence find the set of values of x for which $\left(\frac{x-a}{x-2a}\right)^2 \leqslant \left|\frac{x-a}{x-2a}\right|$. [5]

7 The points A, B, C have position vectors

$$-2\mathbf{i}+2\mathbf{j}-\mathbf{k}, \quad -2\mathbf{i}+\mathbf{j}+2\mathbf{k}, \quad -2\mathbf{j}+\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

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(b) Find the acute angle between the planes OBC and ABC. [4]

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The point D has position vector $t\mathbf{i} - \mathbf{j}$.

(c) Given that the shortest distance between the lines AB and CD is $\sqrt{10}$, find the value of t. [6]

Additional Page

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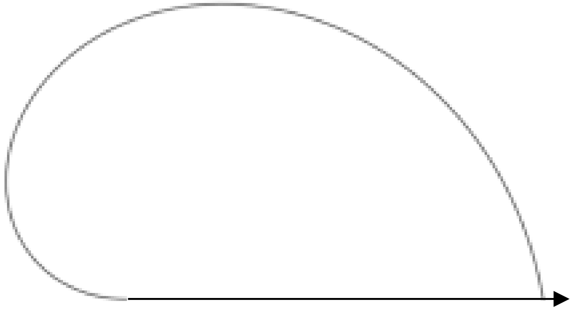
Question	Answer	Marks	Guidance
1(a)	$d=1$	**B1**	
		1	
1(b)	$y=x+1 \Rightarrow x=y-1$	**B1**	Uses correct substitution.
	$y^3+(b-3)y^2+(c-2b+3)y+b-c=0$	**M1 A1**	Substitutes and expands.
	Alternative method for question 1(b)		
	$(\alpha+1)(\beta+1)+(\alpha+1)(\gamma+1)+(\beta+1)(\gamma+1)=c-2b+3$	**B1**	
	$(\alpha+1+\beta+1+\gamma+1)=3-b,\ \ (\alpha+1)(\beta+1)(\gamma+1)=c-b$	**M1**	Using these relationships.
	$y^3+(b-3)y^2+(c-2b+3)y+b-c=0)$	**A1**	
		3	
1(c)	$\beta+1=-(b-3)$	**B1**	Uses sum of roots.
	$-(\alpha+1)(\alpha+1)=c-2b+3$	**B1**	Uses sum of products in pairs.
	$-(\alpha+1)(\beta+1)(\alpha+1)=-(b-c)$	**M1**	Applies product of roots.
	$\Rightarrow (c-2b+3)(b-3)=b-c$	**A1**	AG
		4	

Question	Answer	Marks	Guidance
2	$7^2 - 1 = 48$ is divisible by 12.	**B1**	Checks base case.
	Assume that $7^{2k} - 1$ is divisible by 12 for **some** positive integer k.	**B1**	States inductive hypothesis.
	Then $7^{2k+2} - 1 = 49 \cdot 7^{2k} - 1 = 48 \cdot 7^{2k} + 7^{2k} - 1$ Or $(7^{2k+2} - 1) - (7^{2k} - 1) = 48 \cdot 7^{2k}$	**M1**	Separates $7^{2k} - 1$.
	$7^{2k+2} - 1$ is divisible by 12.	**A1**	
	So if is true for $n = k$ (could be written earlier) it is also true for $n = k + 1$. Hence, by induction, true for every positive integer n.	**A1**	Depends on previous M1 and A1.
		5	

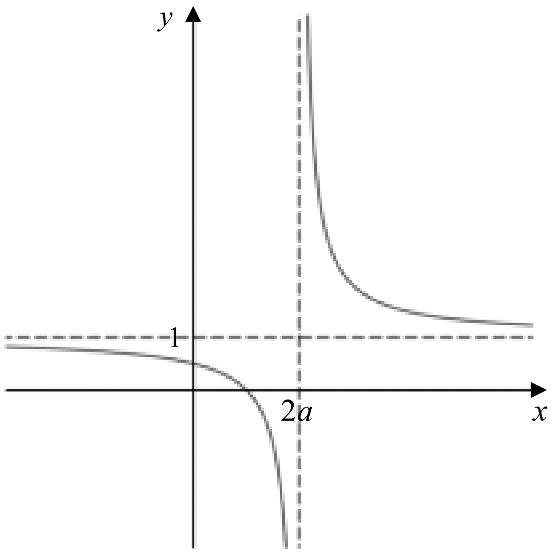
Question	Answer	Marks	Guidance
3(a)	$\left(x^n-\sqrt{\left(x^{2n}+1\right)}\right)\left(x^n+\sqrt{\left(x^{2n}+1\right)}\right)=x^{2n}-\left(x^{2n}+1\right)=-1$	B1	Uses the difference of two squares and obtains the given answer. AG
		1	
3(b)	$u_n=x^{n+1}+\sqrt{\left(x^{2n+2}+1\right)}-x^n-\sqrt{\left(x^{2n}+1\right)}$	B1	Uses the identity given in part **(a)**.
	$\sum_{n=1}^{N}u_n=x^2+\sqrt{\left(x^4+1\right)}-x-\sqrt{\left(x^2+1\right)}+x^3+\sqrt{\left(x^6+1\right)}-x^2-\sqrt{\left(x^4+1\right)}$ $+\ldots+x^{N+1}+\sqrt{\left(x^{2N+2}+1\right)}-x^N-\sqrt{\left(x^{2N}+1\right)}$	M1	Writes at least three complete correct terms, including the first and last.
	$=x^{N+1}+\sqrt{\left(x^{2N+2}+1\right)}-x-\sqrt{\left(x^2+1\right)}$	A1	(Don't allow in terms of n.)
		3	
3(c)	$-1<x<1$	B1	
	$x^{N+1}\to 0$ as $n\to\infty\Rightarrow u_1+u_2+u_3+\ldots=1-x-\sqrt{\left(x^2+1\right)}$	M1 A1	Finds sum to infinity.
		3	

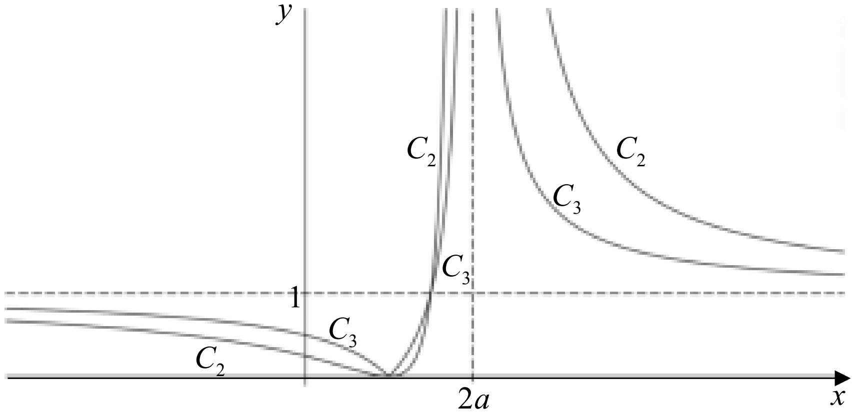
Question	Answer	Marks	Guidance
4(a)	Reflection in the line $y = x$.	**B1**	Only mention reflection (and no other transformation).
		1	
4(b)	Rotation	**B1**	Writes 'rotation' or 'rotate' (and no other transformation)
	$\frac{1}{3}\pi$ **anticlockwise** about the origin.	**B1**	Or 60°
		2	
4(c)	$\mathbf{AB} = \begin{pmatrix} \frac{1}{2}\sqrt{3} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2}\sqrt{3} \end{pmatrix}$ or $\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B}$	**M1**	Finds **AB** or uses product of determinants. Full marks here may be obtained by arguing that both reflection and rotation preserve [absolute] value of area and so also does their combination.
	$\det \mathbf{AB} = -1$	**B1**	
	Area of PQR = |-1| Area of DEF	**A1**	
		3	
4(d)	$(\mathbf{AB})^{-1} = -\begin{pmatrix} -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}\sqrt{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{3} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2}\sqrt{3} \end{pmatrix}$	**M1 A1**	or using $(\mathbf{AB})^{-1} = \mathrm{B}^{-1}\mathrm{A}^{-1}$
		2	

Question	Answer	Marks	Guidance
4(e)	$\begin{pmatrix} \frac{1}{2}\sqrt{3} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2}\sqrt{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x\sqrt{3}+\frac{1}{2}y \\ \frac{1}{2}x-\frac{1}{2}y\sqrt{3} \end{pmatrix}$	B1	Transforms $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} X \\ Y \end{pmatrix}$. Accept $t\begin{pmatrix} 1 \\ m \end{pmatrix}$ instead of $\begin{pmatrix} x \\ y \end{pmatrix}$.
	$\frac{1}{2}-\frac{1}{2}m\sqrt{3}=\frac{1}{2}m\sqrt{3}+\frac{1}{2}m^2$	M1A1	Uses $Y = mX$. OE. Allow working with $y = mx + c$ as long as $c = 0$ is stated explicitly.
	$1-m\sqrt{3}=m\sqrt{3}+m^2 \Rightarrow m^2+2m\sqrt{3}-1=0 \Rightarrow m=\pm 2-\sqrt{3}$	A1	
	$y=(2-\sqrt{3})x$ and $y+(2+\sqrt{3})x=0$	A1	OE
		5	

Question	Answer	Marks	Guidance
5(a)	$\theta = 0$ 	B1	Correct shape, r decreasing.
		B1	Section $\frac{1}{2}\pi \le \theta \le \pi$ correct, becoming tangential at $\theta = \pi$.
	$(\ln(1+\pi), 0)$	B1	May be seen on their diagram. Allow (1.42, 0)
		3	
5(b)	$\frac{1}{2}\int_0^{\pi} \ln^2(1+\pi-\theta)\,d\theta = \frac{1}{2}\int_1^{1+\pi} \ln^2 u\,du$	M1	Uses correct formula with correct limits.
	$= \left[\frac{1}{2}u\ln^2 u\right]_1^{1+\pi} - \int_1^{1+\pi} \ln u\,du$	M1 A1	Integrates by parts once.
	$= \left[\frac{1}{2}u\ln^2 u\right]_1^{1+\pi} - \left(\left[u\ln u\right]_1^{1+\pi} - \int_1^{1+\pi} 1\,du\right)$	M1	Integrates by parts again (or uses known result for integral of ln u)
	$= \left[\frac{1}{2}u\ln^2 u - u\ln u + u\right]_1^{1+\pi}$	A1	
	$\frac{1}{2}(1+\pi)\ln^2(1+\pi) - (1+\pi)\ln(1+\pi) + \pi =$ $\frac{1}{2}(1+\pi)\ln(1+\pi)(\ln(1+\pi) - 2) + \pi$	A1	AG
		6	

Question	Answer	Marks	Guidance
5(c)	$y = \ln(1+\pi-\theta)\sin\theta$	B1	Uses $y = r\sin\theta$
	$\frac{dy}{d\theta} = \ln(1+\pi-\theta)\cos\theta - \frac{\sin\theta}{1+\pi-\theta} = 0$	M1 A1	Sets derivative equal to zero.
	$\cos\theta \neq 0 \Rightarrow (1+\pi-\theta)\ln(1+\pi-\theta) - \tan\theta = 0$	A1	AG
	$(1+\pi-1.2)\ln(1+\pi-1.2) - \tan 1.2 = 0.602$ and $(1+\pi-1.3)\ln(1+\pi-1.3) - \tan 1.3 = -0.634$	B1	Shows sign change. Values correct to 2sf should be shown.
		5	

Question	Answer	Marks	Guidance
6(a)		B1	Asymptotes labelled.
		B1	Correct position and shape. Not too truncated.
		2	
6(b)(i)	$\frac{dy}{dx} = -\frac{2a(x-a)}{(x-2a)^3} = 0$	M1	Sets $\frac{dy}{dx} = 0$.
	Solves for x.	M1	
	Since $x \neq 2a$, stationary point is $(a, 0)$.	A1	
		3	

Question	Answer	Marks	Guidance
6(b)(ii)	$(x-a)^2=(x-a)(x-2a)$ or $(x-a)^2=-(x-a)(x-2a)$	M1	Finds a critical point.
	$x=a$ and $\frac{3}{2}a$ **or** $(a, 0)$ **or** $(\frac{3}{2}a, 1)$	A1	Both x values or one correct point.
	$(a, 0)$ and $(\frac{3}{2}a, 1)$	A1	
		3	
6(c)		B1	Axes and asymptotes.
		B1 FT	C_3 FT from their sketch in part **(a)**. Clearly identified.
		B1	Correct shape of C_2. Clearly identified.
		B1	Relative positions and intersections correct.
	$x \leqslant \frac{3}{2}a$	B1	Accept algebraic method.
		5	

Question	Answer	Marks	Guidance
7(a)	$\overrightarrow{AB} = -\mathbf{j} + 3\mathbf{k} \quad \overrightarrow{AC} = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$	B1	Finds direction vectors of two lines in the plane. $\overrightarrow{BC} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & 3 \\ 1 & -2 & 1 \end{vmatrix} = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}$	M1 A1	Finds normal to the plane *ABC*. OE
	$5(0) + 3(-2) + 1(1) = -5 \Rightarrow 5x + 3y + z = -5$	M1 A1	Substitutes point. OE
		5	
7(b)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 2 \\ 0 & -2 & 1 \end{vmatrix} = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix}$	M1 A1	Finds normal to the plane *OBC*.
	$\begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix} = \sqrt{35}\sqrt{45}\cos\theta$	M1	Uses dot product of normal vectors.
	28.1°	A1	0.49(0) rad
		4	

Question	Answer	Marks	Guidance
7(c)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & 3 \\ t & 1 & -1 \end{vmatrix} = \begin{pmatrix} -2 \\ 3t \\ t \end{pmatrix}$	M1 A1	Finds common perpendicular.
	$\left\lvert \frac{1}{\sqrt{4+10t^2}} \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 3t \\ t \end{pmatrix} \right\rvert = \left\lvert \frac{-4-10t}{\sqrt{4+10t^2}} \right\rvert$	M1 A1	Uses formula for perpendicular distance.
	$\left\lvert \frac{-4-10t}{\sqrt{4+10t^2}} \right\rvert = \sqrt{10} \Rightarrow (4+10t)^2 = 10\left(4+10t^2\right)$	M1	Sets equal to $\sqrt{10}$ and solves for t.
	$t = \frac{3}{10}$	A1	
		6	

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FURTHER MATHEMATICS **9231/11**

Paper 1 Further Pure Mathematics 1 **May/June 2020**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS
- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION
- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Blank pages are indicated.

DC (PQ) 197982

1 Let a be a positive constant.

(a) Sketch the curve with equation $y = \frac{ax}{x+7}$. [2]

(b) Sketch the curve with equation $y = \left|\dfrac{ax}{x+7}\right|$ and find the set of values of x for which $\left|\dfrac{ax}{x+7}\right| > \dfrac{a}{2}$. [4]

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2 The cubic equation $6x^3 + px^2 - 3x - 5 = 0$, where p is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are α^2, β^2, γ^2. [3]

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(b) It is given that $\alpha^2 + \beta^2 + \gamma^2 = 2(\alpha + \beta + \gamma)$.

(i) Find the value of p. [3]

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(ii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. [2]

3 The curve C has equation $y = \dfrac{x^2}{2x+1}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Find the coordinates of the stationary points on C. [3]

(c) Sketch C. [3]

4 **(a)** By first expressing $\frac{1}{r^2-1}$ in partial fractions, show that

$$\sum_{r=2}^{n} \frac{1}{r^2-1} = \frac{3}{4} - \frac{an+b}{2n(n+1)},$$

where a and b are integers to be found. [5]

(b) Deduce the value of $\sum_{r=2}^{\infty} \frac{1}{r^2 - 1}$. [1]

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(c) Find the limit, as $n \to \infty$, of $\sum_{r=n+1}^{2n} \frac{n}{r^2 - 1}$. [4]

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5 The lines l_1 and l_2 have equations $\mathbf{r} = 3\mathbf{i} + 3\mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} + 4\mathbf{k})$ and $\mathbf{r} = 3\mathbf{i} - 5\mathbf{j} - 6\mathbf{k} + \mu(5\mathbf{j} + 6\mathbf{k})$ respectively.

(a) Find the shortest distance between l_1 and l_2. [5]

The plane Π contains l_1 and is parallel to the vector $\mathbf{i}+\mathbf{k}$.

(b) Find the equation of Π, giving your answer in the form $ax+by+cz=d$. [4]

(c) Find the acute angle between l_2 and Π. [3]

6 Let $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$.

(a) The transformation in the x-y plane represented by $\mathbf{A}^{-1}$ transforms a triangle of area $30\,\text{cm}^2$ into a triangle of area $d\,\text{cm}^2$.

Find the value of d. [3]

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(b) Prove by mathematical induction that, for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}.$$ [5]

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(c) The line $y = 2x$ is invariant under the transformation in the x-y plane represented by $\mathbf{A}^n\mathbf{B}$, where $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 33 & 0 \end{pmatrix}$.

Find the value of n. [5]

7 The curve C_1 has polar equation $r = \theta\cos\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) The point on C_1 furthest from the line $\theta = \frac{1}{2}\pi$ is denoted by P. Show that, at P,

$$2\theta\tan\theta - 1 = 0$$

and verify that this equation has a root between 0.6 and 0.7. [5]

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The curve C_2 has polar equation $r = \theta\sin\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. The curves C_1 and C_2 intersect at the pole, denoted by O, and at another point Q.

(b) Find the polar coordinates of Q, giving your answers in exact form. [2]

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(c) Sketch C_1 and C_2 on the same diagram. [3]

(d) Find, in terms of π, the area of the region bounded by the arc OQ of C_1 and the arc OQ of C_2. [7]

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FURTHER MATHEMATICS **9231/11**

Paper 1 Further Pure Mathematics 1 **May/June 2020**

MARK SCHEME

Maximum Mark: 75

Published

Students did not sit exam papers in the June 2020 series due to the Covid-19 global pandemic.

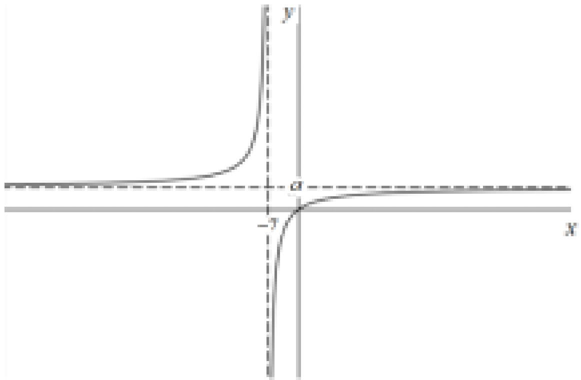
This mark scheme is published to support teachers and students and should be read together with the question paper. It shows the requirements of the exam. The answer column of the mark scheme shows the proposed basis on which Examiners would award marks for this exam. Where appropriate, this column also provides the most likely acceptable alternative responses expected from students. Examiners usually review the mark scheme after they have seen student responses and update the mark scheme if appropriate. In the June series, Examiners were unable to consider the acceptability of alternative responses, as there were no student responses to consider.

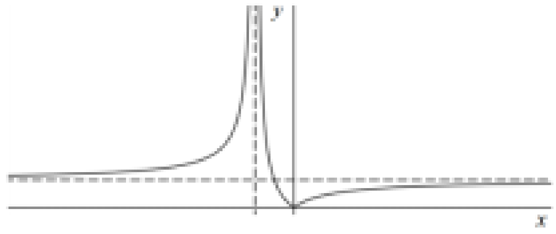
Mark schemes should usually be read together with the Principal Examiner Report for Teachers. However, because students did not sit exam papers, there is no Principal Examiner Report for Teachers for the June 2020 series.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the June 2020 series for most Cambridge IGCSE™ and Cambridge International A & AS Level components, and some Cambridge O Level components.

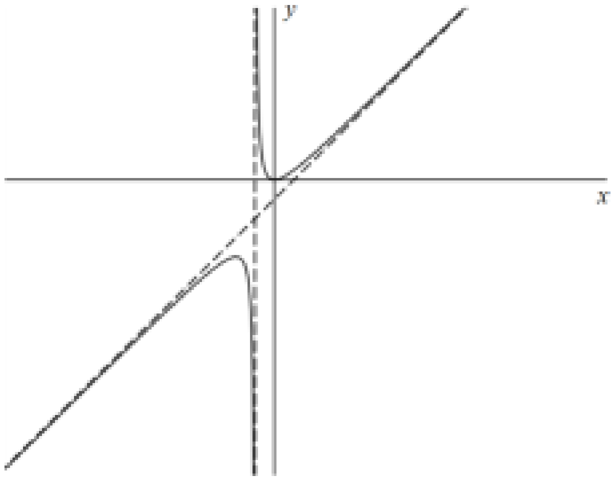
This document consists of **15** printed pages.

Question	Answer	Marks
1(a)		B1
		B1
	(**B1** for axes and asymptotes correct, **B1** for branches correct)	
		2

Question	Answer	Marks
1(b)	 (**B1 FT** from sketch in part **(a)**)	**B1 FT**
	$\frac{ax}{x+7}=\frac{a}{2}$ or $\frac{ax}{x+7}=-\frac{a}{2}$	**M1**
	$x=7$ and $x=-\frac{7}{3}$	**A1**
	$x<-7$, $-7<x<-\frac{7}{3}$, $x>7$	**A1**
		4

Question	Answer	Marks
2(a)	$y = x^2$	B1
	$6y^{\frac{3}{2}} + py - 3y^{\frac{1}{2}} - 5 = 0 \Rightarrow y^{\frac{1}{2}}(6y-3) = -py + 5$ $y(6y-3)^2 = (-py+5)^2 \Rightarrow y(36y^2 - 36y + 9) = p^2y^2 - 10py + 25$	M1
	$36y^3 - (p^2+36)y^2 + (10p+9)y - 25 = 0$	A1
		3
2(b)(i)	$\alpha^2 + \beta^2 + \gamma^2 = \dfrac{p^2+36}{36}$	B1
	$\dfrac{p^2+36}{36} = -\dfrac{2p}{6} \Rightarrow p^2 + 12p + 36 = 0$	M1
	$p = -6$	A1
		3
2(b)(ii)	$6(\alpha^3+\beta^3+\gamma^3) = 6(\alpha^2+\beta^2+\gamma^2) + 3(\alpha+\beta+\gamma) + 15$	M1
	$\alpha^3 + \beta^3 + \gamma^3 = 5$	A1
		2

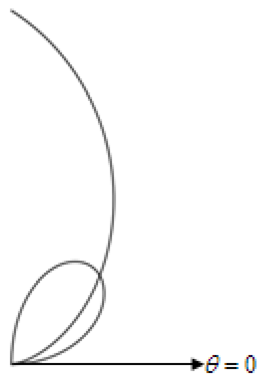
Question	Answer	Marks
3(a)	$x = -\frac{1}{2}$	B1
	$y = \frac{(2x+1)(\frac{1}{2}x - \frac{1}{4}) + \frac{1}{4}}{2x+1} = \frac{1}{2}x - \frac{1}{4} + \frac{1}{4(2x+1)}$	M1
	$y = \frac{1}{2}x - \frac{1}{4}$	A1
		3
3(b)	$\frac{dy}{dx} = \frac{1}{2} - \frac{1}{2(2x+1)^2} = 0 \Rightarrow (2x+1)^2 = 1$	M1
	$x = 0, -1$	A1
	$(0,0), (-1,-1)$	A1
		3

Question	Answer	Marks
3(c)		B1
		B1
	(**B1** for axes and asymptotes correct, **B1** for upper branch correct, **B1** for lower branch correct)	B1
		3

Question	Answer	Marks
4(a)	$\frac{1}{r^2-1}=\frac{1}{(r-1)(r+1)}=\frac{1}{2}\left(\frac{1}{r-1}-\frac{1}{r+1}\right)$	M1 A1
	$\sum_{r=2}^{n}\frac{1}{r^2-1}=\frac{1}{2}\left(\frac{1}{1}-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+\frac{1}{4}-\frac{1}{6}+\cdots+\frac{1}{n-2}-\frac{1}{n}+\frac{1}{n-1}-\frac{1}{n+1}\right)$	M1
	$=\frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}\right)$	A1
	$=\frac{3}{4}-\frac{2n+1}{2n(n+1)}$	A1
		5
4(b)	$\frac{3}{4}$	B1
		1
4(c)	$\sum_{r=n+1}^{2n}\frac{n}{r^2-1}=\sum_{r=2}^{2n}\frac{n}{r^2-1}-\sum_{r=2}^{n}\frac{n}{r^2-1}$	M1
	$=n\left(\frac{3}{4}-\frac{4n+1}{4n(2n+1)}\right)-n\left(\frac{3}{4}-\frac{2n+1}{2n(n+1)}\right)=\frac{n(2n+1)}{2n(n+1)}-\frac{n(4n+1)}{4n(2n+1)}$	M1A1
	$\to\frac{1}{2}$ as $n\to\infty$	A1
		4

Question	Answer	Marks
5(a)	$\begin{pmatrix}3\\-5\\-6\end{pmatrix}-\begin{pmatrix}3\\0\\3\end{pmatrix}=\begin{pmatrix}0\\-5\\-9\end{pmatrix}$	B1
	$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\1 & 4 & 4\\0 & 5 & 6\end{vmatrix}=\begin{pmatrix}4\\-6\\5\end{pmatrix}$	M1 A1
	$\frac{1}{\sqrt{77}}\left\|\begin{pmatrix}0\\-5\\-9\end{pmatrix}\bullet\begin{pmatrix}4\\-6\\5\end{pmatrix}\right\|=\frac{15}{\sqrt{77}}=1.71$	M1 A1
		5
5(b)	$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\1 & 4 & 4\\1 & 0 & 1\end{vmatrix}=\begin{pmatrix}4\\3\\-4\end{pmatrix}$	M1 A1
	$4(3)+3(0)-4(3)=0 \Rightarrow 4x+3y-4z=0$	M1 A1
		4
5(c)	$\begin{pmatrix}0\\5\\6\end{pmatrix}\bullet\begin{pmatrix}4\\3\\-4\end{pmatrix}=\sqrt{61}\sqrt{41}\cos\alpha \Rightarrow \cos\alpha=\frac{-9}{\sqrt{61}\sqrt{41}}$ (**A1 FT** *their* normal)	M1 A1FT
	Acute angle between l_2 and Π is $\alpha-90=10.4^\circ$	A1
		3

Question	Answer	Marks
6(a)	$\det \mathbf{A}^{-1} = (\det \mathbf{A})^{-1} = \frac{1}{2}$	M1 A1
	$d = 15$	A1
		3
6(b)	$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2^1 & 0 \\ 2^1 - 1 & 1 \end{pmatrix}$ so true when $n = 1$.	B1
	Assume that it is true for $n = k$, so $\mathbf{A}^k = \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix}$.	B1
	Then $\mathbf{A}^{k+1} = \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2^{k+1} & 0 \\ 2(2^k - 1) + 1 & 1 \end{pmatrix} = \begin{pmatrix} 2^{k+1} & 0 \\ 2^{k+1} - 1 & 1 \end{pmatrix}$	M1A1
	So, it is also true for $n = k + 1$. Hence, by induction, true for all positive integers.	A1
		5
6(c)	$\mathbf{A}^n\mathbf{B} = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 33 & 0 \end{pmatrix} = \begin{pmatrix} 2^n & 0 \\ 2^n + 32 & 0 \end{pmatrix}$	M1A1
	$\begin{pmatrix} 2^n & 0 \\ 2^n + 32 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2^n x \\ (2^n + 32)x \end{pmatrix}$	B1
	$(2^n + 32)x = 2^{n+1}x \Rightarrow 2^n = 32 \Rightarrow n = 5$	M1 A1
		5

Question	Answer	Marks
7(a)	$x = \theta\cos^2\theta$	B1
	$\frac{dx}{d\theta} = -2\theta\cos\theta\sin\theta + \cos^2\theta = 0$	M1 A1
	$\cos\theta \neq 0 \Rightarrow 2\theta\tan\theta - 1 = 0$	A1
	$2(0.6)\tan 0.6 - 1 = -0.179$ and $2(0.7)\tan 0.7 - 1 = 0.179$	B1
		5
7(b)	$\theta\cos\theta = \theta\sin\theta \Rightarrow \tan\theta = 1$	M1
	$\left(\frac{1}{8}\pi\sqrt{2}, \frac{1}{4}\pi\right)$	A1
		2
7(c)	 (**B1** for initial line drawn and C_1 correct, **B1** for C_2 correct, **B1** for intersection correct)	B1 B1 B1
		3

Question	Answer	Marks
7(d)	$\frac{1}{2}\int_0^{\frac{1}{4}\pi}\theta^2\left(\cos^2\theta-\sin^2\theta\right)\mathrm{d}\theta$	M1
	$=\frac{1}{2}\int_0^{\frac{1}{4}\pi}\theta^2\cos 2\theta\,\mathrm{d}\theta$	M1
	$=\frac{1}{2}\left(\left[\frac{1}{2}\theta^2\sin 2\theta\right]_0^{\frac{1}{4}\pi}-\int_0^{\frac{1}{4}\pi}\theta\sin 2\theta\,\mathrm{d}\theta\right)$	M1 A1
	$=\frac{1}{2}\left(\left[\frac{1}{2}\theta^2\sin 2\theta\right]_0^{\frac{1}{4}\pi}-\left[-\frac{1}{2}\theta\cos 2\theta\right]_0^{\frac{1}{4}\pi}-\frac{1}{2}\int_0^{\frac{1}{4}\pi}\cos 2\theta\,\mathrm{d}\theta\right)$	M1
	$=\frac{1}{4}\left[\theta^2\sin 2\theta+\theta\cos 2\theta-\frac{1}{2}\sin 2\theta\right]_0^{\frac{1}{4}\pi}$	A1
	$=\frac{1}{64}\pi^2-\frac{1}{8}$	A1
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Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME			
CENTRE NUMBER		CANDIDATE NUMBER	

2388089987

FURTHER MATHEMATICS **9231/13**

Paper 1 Further Pure Mathematics 1 **May/June 2020**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Blank pages are indicated.

DC (LK) 187467/2

1 The cubic equation $7x^3 + 3x^2 + 5x + 1 = 0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$. [3]

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(b) Find the value of $\alpha^{-2} + \beta^{-2} + \gamma^{-2}$. [2]

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(c) Find the value of $\alpha^{-3} + \beta^{-3} + \gamma^{-3}$. [2]

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2 The sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = 1$ and $u_{n+1} = 2u_n + 1$ for $n \geqslant 1$.

(a) Prove by induction that $u_n = 2^n - 1$ for all positive integers n. [5]

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(b) Deduce that u_{2n} is divisible by u_n for $n \geqslant 1$. [2]

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3 Let $S_n = 2^2 + 6^2 + 10^2 + \ldots + (4n-2)^2$.

(a) Use standard results from the List of Formulae (MF19) to show that $S_n = \frac{4}{3}n(4n^2 - 1)$. [4]

(b) Express $\frac{n}{S_n}$ in partial fractions and find $\sum_{n=1}^{N} \frac{n}{S_n}$ in terms of N. [4]

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(c) Deduce the value of $\sum_{n=1}^{\infty} \frac{n}{S_n}$. [1]

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4 The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} k & 0 & 2 \\ 0 & -1 & -1 \\ 1 & 1 & -k \end{pmatrix},$$

where k is a real constant.

(a) Show that **A** is non-singular. [3]

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The matrices **B** and **C** are given by

$$\mathbf{B} = \begin{pmatrix} 0 & -3 \\ -1 & 3 \\ 0 & 0 \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} -3 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

It is given that $\mathbf{CAB} = \begin{pmatrix} -2 & -\frac{3}{2} \\ -1 & -\frac{3}{2} \end{pmatrix}$.

(b) Find the value of k. [3]

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(c) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **CAB**. [5]

5 The curve C has polar equation $r = a\tan\theta$, where a is a positive constant and $0 \leqslant \theta \leqslant \frac{1}{4}\pi$.

(a) Sketch C and state the greatest distance of a point on C from the pole. [2]

(b) Find the exact value of the area of the region bounded by C and the half-line $\theta = \frac{1}{4}\pi$. [4]

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(c) Show that C has Cartesian equation $y = \dfrac{x^2}{\sqrt{a^2 - x^2}}$. [3]

(d) Using your answer to part **(b)**, deduce the exact value of $\displaystyle\int_0^{\frac{1}{2}a\sqrt{2}} \frac{x^2}{\sqrt{a^2 - x^2}}\,dx$. [2]

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6 The curve C has equation $y = \dfrac{10 + x - 2x^2}{2x - 3}$.

(a) Find the equations of the asymptotes of C. [3]

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(b) Show that C has no turning points. [3]

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(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left|\dfrac{10+x-2x^2}{2x-3}\right|$ and find in exact form the set of values of x for which $\left|\dfrac{10+x-2x^2}{2x-3}\right| < 4$. [6]

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7 The lines l_1 and l_2 have equations $\mathbf{r} = -5\mathbf{j} + \lambda(5\mathbf{i} + 2\mathbf{k})$ and $\mathbf{r} = 4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} + \mu(\mathbf{j} + \mathbf{k})$ respectively. The plane Π contains l_1 and is parallel to l_2.

(a) Find the equation of Π, giving your answer in the form $ax + by + cz = d$. [4]

(b) Find the distance between l_2 and Π. [3]

The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2.

(c) Show that P has position vector $\frac{55}{27}\mathbf{i} - 5\mathbf{j} + \frac{22}{27}\mathbf{k}$ and state a vector equation for PQ. [8]

Cambridge Assessment International Education

Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/13**

Paper 1 Further Pure Mathematics 1 **May/June 2020**

MARK SCHEME

Maximum Mark: 75

Published

Students did not sit exam papers in the June 2020 series due to the Covid-19 global pandemic.

This mark scheme is published to support teachers and students and should be read together with the question paper. It shows the requirements of the exam. The answer column of the mark scheme shows the proposed basis on which Examiners would award marks for this exam. Where appropriate, this column also provides the most likely acceptable alternative responses expected from students. Examiners usually review the mark scheme after they have seen student responses and update the mark scheme if appropriate. In the June series, Examiners were unable to consider the acceptability of alternative responses, as there were no student responses to consider.

Mark schemes should usually be read together with the Principal Examiner Report for Teachers. However, because students did not sit exam papers, there is no Principal Examiner Report for Teachers for the June 2020 series.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the June 2020 series for most Cambridge IGCSE™ and Cambridge International A & AS Level components, and some Cambridge O Level components.

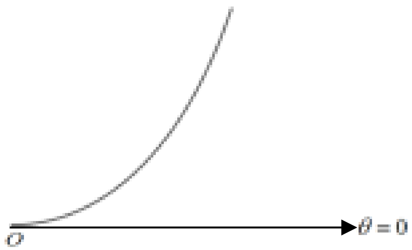
This document consists of **14** printed pages.

Question	Answer	Marks
1(a)	$y = x^{-1}$	B1
	$7y^{-3} + 3y^{-2} + 5y^{-1} + 1 = 0 \Rightarrow y^3 + 5y^2 + 3y + 7 = 0$	M1 A1
		3
1(b)	$\alpha^{-2} + \beta^{-2} + \gamma^{-2} = (-5)^2 - 2(3) = 19$	M1 A1
1(c)	$\alpha^{-3} + \beta^{-3} + \gamma^{-3} = -5(19) - 3(-5) - 21 = -101$	M1 A1
		4

Question	Answer	Marks
2(a)	$u_1 = 1 = 2^1 - 1$	B1
	Assume that it is true for $n = k$, so $u_k = 2^k - 1$.	B1
	Then $u_{k+1} = 2(2^k - 1) + 1 = 2^{k+1} - 1$	M1 A1
	So, it is also true for $n = k + 1$. Hence, by induction, true for all positive integers.	A1
		5
2(b)	$\frac{u_{2n}}{u_n} = \frac{2^{2n} - 1}{2^n - 1} = \frac{(2^n - 1)(2^n + 1)}{(2^n - 1)} = 2^n + 1$	M1 A1
		2

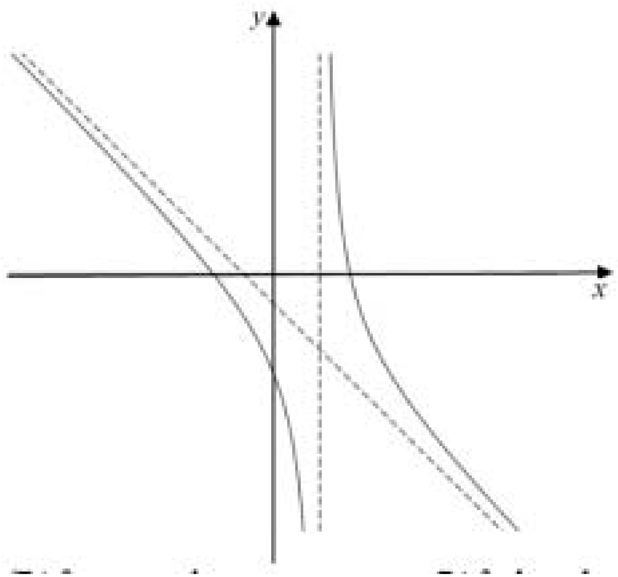
Question	Answer	Marks
3(a)	$S_n = \sum_{r=1}^{n}(4r-2)^2 = 16\sum_{r=1}^{n} r^2 - 16\sum_{r=1}^{n} r + 4n$	M1 A1
	$= \frac{8}{3}n(n+1)(2n+1) - 8n(n+1) + 4n$	M1
	$\frac{4}{3}n(4n^2+6n+2-6n-6+3) = \frac{4}{3}n\left(4n^2-1\right)$ **AG**	A1
		4
3(b)	$\frac{n}{S_n} = \frac{3}{4(2n-1)(2n+1)} = \frac{3}{8}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$	M1 A1
	$\sum_{n=1}^{N}\frac{n}{S_n} = \frac{3}{8}\left(1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} \ldots + \frac{1}{2N-1} - \frac{1}{2N+1}\right)$	M1
	$= \frac{3}{8}\left(1 - \frac{1}{2N+1}\right)$	A1
		4
3(c)	$\frac{3}{8}$	B1
		1

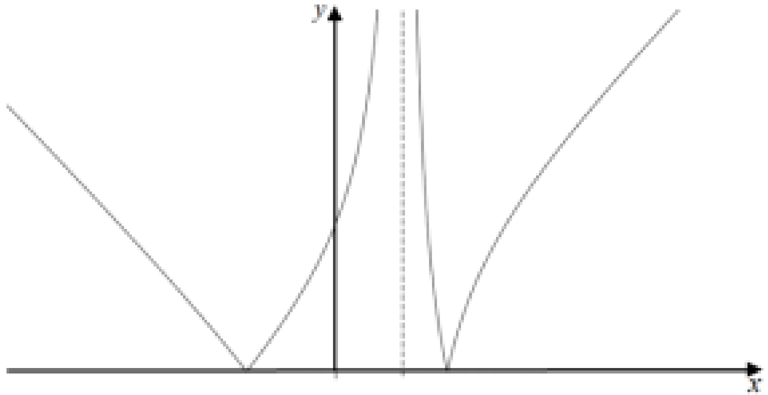
Question	Answer	Marks
4(a)	$k\begin{vmatrix}-1 & -1\\ 1 & -k\end{vmatrix}+2\begin{vmatrix}0 & -1\\ 1 & 1\end{vmatrix}=0\Rightarrow k^2+k+2=0$	M1 A1
	$1-4(2)=-7<0\Rightarrow$ Non-singular	A1
		3
4(b)	$\begin{pmatrix}-3 & -1 & 1\\ 1 & 1 & 2\end{pmatrix}\begin{pmatrix}k & 0 & 2\\ 0 & -1 & -1\\ 1 & 1 & -k\end{pmatrix}\begin{pmatrix}0 & -3\\ -1 & 3\\ 0 & 0\end{pmatrix}=\begin{pmatrix}-3 & -1 & 1\\ 1 & 1 & 2\end{pmatrix}\begin{pmatrix}0 & -3k\\ 1 & -3\\ -1 & 0\end{pmatrix}$	M1 A1
	$=\begin{pmatrix}-2 & 9k+3\\ -1 & -3k-3\end{pmatrix}=\begin{pmatrix}-2 & -\frac{3}{2}\\ -1 & -\frac{3}{2}\end{pmatrix}\Rightarrow k=-\frac{1}{2}$	A1
		3
4(c)	$\begin{pmatrix}2 & \frac{3}{2}\\ 1 & \frac{3}{2}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}2x+\frac{3}{2}y\\ x+\frac{3}{2}y\end{pmatrix}$	B1
	$x+\frac{3}{2}mx=m\left(2x+\frac{3}{2}mx\right)$	M1 A1
	$1+\frac{3}{2}m=2m+\frac{3}{2}m^2\Rightarrow 3m^2+m-2=0$	A1
	$y=-x$ and $3y-2x=0$	A1
		5

Question	Answer	Marks
5(a)		B1
	Maximum distance of C from the pole is a.	B1
		2
5(b)	$\frac{1}{2}a^2\int_0^{\frac{1}{4}\pi}\tan^2\theta\,d\theta$	M1
	$=\frac{1}{2}a^2\int_0^{\frac{1}{4}\pi}\sec^2\theta-1\,d\theta=\frac{1}{2}a^2\left[\tan\theta-\theta\right]_0^{\frac{1}{4}\pi}$	M1 A1
	$=\frac{1}{2}a^2\left(1-\frac{1}{4}\pi\right)$	A1
		4
5(c)	$\sqrt{x^2+y^2}=a\frac{y}{x}$	B1
	$x^2\left(x^2+y^2\right)=a^2y^2\Rightarrow y^2\left(a^2-x^2\right)=x^4$	M1
	$y=\frac{x^2}{\sqrt{\left(a^2-x^2\right)}}$ **AG**	A1
		3

Question	Answer	Marks
5(d)	$\frac{1}{2}\left(a\cos\frac{1}{4}\pi\right)\left(a\sin\frac{1}{4}\pi\right)-\frac{1}{2}a^2\left(1-\frac{1}{4}\pi\right)=\frac{1}{4}a^2\left(\frac{1}{2}\pi-1\right)$ **(A1 FT** *their* **(b))**	**M1 A1 FT**
		11

Question	Answer	Marks
6(a)	$x=\frac{3}{2}$	**B1**
	$-2x^2+x+10=(2x-3)(-x-1)+7 \Rightarrow y=-x-1$	**M1 A1**
		3
6(b)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(2x-3)(1-4x)-2\left(10+x-2x^2\right)}{(2x-3)^2}$	**M1**
	$4x^2-12x+23=0 \quad \left(\text{or } \frac{\mathrm{d}y}{\mathrm{d}x}=-1-\frac{14}{(2x-3)^2}\right)$	**A1**
	$12^2-4(4)(23)=-224<0$ (or $y'<0$) $\Rightarrow$ No turning points.	**A1**
		3

Question	Answer	Marks
6(c)	 (**B1** for axes and asymptotes correct, **B1** for branches correct)	**B1** **B1**
	$(0,-\frac{10}{3}),(-2,0),(\frac{5}{2},0)$	**B1**
		3

Question	Answer	Marks
6(d)	 (**B1 FT** for their sketch in **(c)**, **B1** for correct shape at infinity)	**B1 FT** **B1**
	$\frac{10+x-2x^2}{2x-3}=4$ or $\frac{10+x-2x^2}{2x-3}=-4$ $2x^2+7x-22=0$ or $2x^2-9x+2=0$	**M2**
	$x=-\frac{11}{2},2$ or $x=\frac{1}{4}\left(9\pm\sqrt{65}\right)$	**A1**
	$-\frac{11}{2}<x<\frac{1}{4}\left(9-\sqrt{65}\right)$ and $2<x<\frac{1}{4}\left(9+\sqrt{65}\right)$	**A1 FT**
		6

Question	Answer	Marks
7(a)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix} = \begin{pmatrix} -2 \\ -5 \\ 5 \end{pmatrix}$	M1 A1
	$-5(-5) = 25$	M1
	$-2x - 5y + 5z = 25$	A1
		4
7(b)	$\begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ -7 \\ 2 \end{pmatrix}$	M1
	$\frac{1}{\sqrt{54}} \left\| \begin{pmatrix} -4 \\ -7 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -5 \\ 5 \end{pmatrix} \right\| = \frac{53}{\sqrt{54}} = 7.21$	M1 A1
		3

Question	Answer	Marks
7(c)	$\overrightarrow{OP}=\begin{pmatrix}5\lambda\\-5\\2\lambda\end{pmatrix}, \overrightarrow{OQ}=\begin{pmatrix}4\\2+\mu\\-2+\mu\end{pmatrix} \Rightarrow \overrightarrow{PQ}=\begin{pmatrix}4-5\lambda\\7+\mu\\-2+\mu-2\lambda\end{pmatrix}$	**M1 A1**
	$\begin{pmatrix}4-5\lambda\\7+\mu\\-2+\mu-2\lambda\end{pmatrix}\bullet\begin{pmatrix}5\\0\\2\end{pmatrix}=0$ or $\begin{pmatrix}4-5\lambda\\7+\mu\\-2+\mu-2\lambda\end{pmatrix}=k\begin{pmatrix}-2\\-5\\5\end{pmatrix}$	**M1**
	$-29\lambda+2\mu=-16$	**A1**
	$\begin{pmatrix}4-5\lambda\\7+\mu\\-2+\mu-2\lambda\end{pmatrix}\bullet\begin{pmatrix}0\\1\\1\end{pmatrix}=0\Rightarrow -2\lambda+2\mu=-5$	**A1**
	$\lambda=\frac{11}{27}\Rightarrow \overrightarrow{OP}=\frac{1}{27}\begin{pmatrix}55\\-135\\22\end{pmatrix}=\frac{55}{27}\mathbf{i}-5\mathbf{j}+\frac{22}{27}\mathbf{k}$	**M1 A1**
	$\mathbf{r}=\frac{1}{27}\begin{pmatrix}55\\-135\\22\end{pmatrix}+k\begin{pmatrix}-2\\-5\\5\end{pmatrix}$ (**B1 FT** their common perpendicular)	**B1 FT**
		8