Cambridge International A Level

Cambridge Assessment International Education
Cambridge International Advanced Level

CANDIDATE NAME

CENTRE NUMBER | CANDIDATE NUMBER

5418419037

FURTHER MATHEMATICS **9231/11**

Paper 1 **October/November 2019**

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF10)

READ THESE INSTRUCTIONS FIRST

Write your centre number, candidate number and name in the spaces at the top of this page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.
DO **NOT** WRITE IN ANY BARCODES.

Answer **all** the questions in the space provided. If additional space is required, you should use the lined page at the end of this booklet. The question number(s) must be clearly shown.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **22** printed pages and **2** blank pages.

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1 The curve C has equation $y = x^a$ for $0 \leqslant x \leqslant 1$, where a is a positive constant. Find, in terms of a, the coordinates of the centroid of the region enclosed by C, the line $x = 1$ and the x-axis. [6]

2 It is given that $y = \ln(ax + 1)$, where a is a positive constant. Prove by mathematical induction that, for every positive integer n,

$$\frac{d^n y}{dx^n} = (-1)^{n-1}\frac{(n-1)!a^n}{(ax+1)^n}.$$

[6]

3 The integral I_n, where n is a positive integer, is defined by

$$I_n = \int_{\frac{1}{2}}^{1} x^{-n} \sin \pi x \, dx.$$

(i) Show that

$$n(n+1)I_{n+2} = 2^{n+1}n + \pi - \pi^2 I_n.$$ [5]

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(ii) Find I_5 in terms of π and I_1. [2]

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4 The line $y = 2x + 1$ is an asymptote of the curve C with equation

$$y = \frac{x^2 + 1}{ax + b}.$$

(i) Find the values of the constants a and b. [3]

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(ii) State the equation of the other asymptote of C. [1]

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(iii) Sketch C. [Your sketch should indicate the coordinates of any points of intersection with the y-axis. You do not need to find the coordinates of any stationary points.] [3]

5 Let $S_N = \sum_{r=1}^{N}(5r+1)(5r+6)$ and $T_N = \sum_{r=1}^{N}\frac{1}{(5r+1)(5r+6)}$.

(i) Use standard results from the List of Formulae (MF10) to show that

$$S_N = \tfrac{1}{3}N(25N^2 + 90N + 83).$$ [3]

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(ii) Use the method of differences to express T_N in terms of N. [4]

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(iii) Find $\lim_{N\to\infty}(N^{-3}S_N T_N)$. [2]

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6 With O as the origin, the points A, B, C have position vectors

$$\mathbf{i} - \mathbf{j}, \quad 2\mathbf{i} + \mathbf{j} + 7\mathbf{k}, \quad \mathbf{i} - \mathbf{j} + \mathbf{k}$$

respectively.

(i) Find the shortest distance between the lines OC and AB. [5]

(ii) Find the cartesian equation of the plane containing the line OC and the common perpendicular of the lines OC and AB. [4]

7 The equation $x^3 + 2x^2 + x + 7 = 0$ has roots α, β, γ.

(i) Use the relation $x^2 = -7y$ to show that the equation

$$49y^3 + 14y^2 - 27y + 7 = 0$$

has roots $\frac{\alpha}{\beta\gamma}, \frac{\beta}{\gamma\alpha}, \frac{\gamma}{\alpha\beta}$. [4]

(ii) Show that $\dfrac{\alpha^2}{\beta^2\gamma^2} + \dfrac{\beta^2}{\gamma^2\alpha^2} + \dfrac{\gamma^2}{\alpha^2\beta^2} = \dfrac{58}{49}$. [3]

(iii) Find the exact value of $\dfrac{\alpha^3}{\beta^3\gamma^3} + \dfrac{\beta^3}{\gamma^3\alpha^3} + \dfrac{\gamma^3}{\alpha^3\beta^3}$. [2]

8 The matrix **M** is defined by

$$\mathbf{M} = \begin{pmatrix} 2 & m & 1 \\ 0 & m & 7 \\ 0 & 0 & 1 \end{pmatrix},$$

where $m \neq 0, 1, 2$.

(i) Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{M} = \mathbf{PDP}^{-1}$. [7]

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(ii) Find $\mathbf{M}^7\mathbf{P}$. [3]

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9 **(i)** Use de Moivre's theorem to show that

$$\sec 6\theta = \frac{\sec^6 \theta}{32 - 48\sec^2 \theta + 18\sec^4 \theta - \sec^6 \theta}.$$ [6]

(ii) Hence obtain the roots of the equation

$$3x^6 - 36x^4 + 96x^2 - 64 = 0$$

in the form sec $q\pi$, where q is rational. [5]

10 The matrix $\mathbf{A}$ is defined by

$$\mathbf{A} = \begin{pmatrix} 1 & 5 & 1 \\ 1 & -2 & -2 \\ 2 & 3 & \theta \end{pmatrix}.$$

(i) **(a)** Find the rank of $\mathbf{A}$ when $\theta \neq -1$. [3]

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(b) Find the rank of $\mathbf{A}$ when $\theta = -1$. [1]

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Consider the system of equations

$$\begin{aligned} x + 5y + z &= -1, \\ x - 2y - 2z &= 0, \\ 2x + 3y + \theta z &= \theta. \end{aligned}$$

(ii) Solve the system of equations when $\theta \neq -1$. [3]

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(iii) Find the general solution when $\theta = -1$. [3]

(iv) Show that if $\theta = -1$ and $\phi \neq -1$ then $\mathbf{Ax} = \begin{pmatrix} -1 \\ 0 \\ \phi \end{pmatrix}$ has no solution. [2]

11 Answer only **one** of the following two alternatives.

EITHER

It is given that $w = \cos y$ and

$$\tan y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 2\tan y \frac{dy}{dx} = 1 + e^{-2x} \sec y.$$

(i) Show that

$$\frac{d^2w}{dx^2} + 2\frac{dw}{dx} + w = -e^{-2x}.$$ [4]

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(ii) Find the particular solution for y in terms of x, given that when $x = 0$, $y = \frac{1}{3}\pi$ and $\frac{dy}{dx} = \frac{1}{\sqrt{3}}$. [10]

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OR

The curves C_1 and C_2 have polar equations, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$, as follows:

$$C_1 : r = 2(e^{\theta} + e^{-\theta}),$$
$$C_2 : r = e^{2\theta} - e^{-2\theta}.$$

The curves intersect at the point P where $\theta = \alpha$.

(i) Show that $e^{2\alpha} - 2e^{\alpha} - 1 = 0$. Hence find the exact value of α and show that the value of r at P is $4\sqrt{2}$. [6]

(ii) Sketch C_1 and C_2 on the same diagram. [3]

(iii) Find the area of the region enclosed by C_1, C_2 and the initial line, giving your answer correct to 3 significant figures. [5]

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Additional Page

If you use the following lined page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.

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MARK SCHEME

Maximum Mark: 100

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

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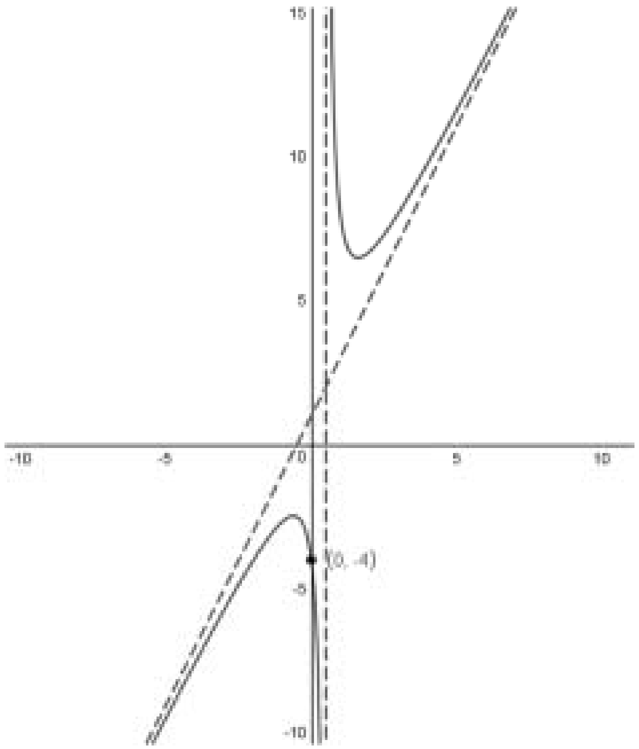
Question	Answer	Marks	Guidance
1	$A=\int_0^1 x^a\,\mathrm{d}x=\left[\frac{x^{a+1}}{a+1}\right]_0^1=\frac{1}{a+1}$	B1	Finds area of region.
	$A\bar{x}=\int_0^1 xy\,\mathrm{d}x=\int_0^1 x^{a+1}\,\mathrm{d}x=\left[\frac{x^{a+2}}{a+2}\right]_0^1=\frac{1}{a+2}$	M1 A1	Finds $\int_0^1 xy\,\mathrm{d}x$.
	$2A\bar{y}=\int_0^1 y^2\,\mathrm{d}x=\int_0^1 x^{2a}\,\mathrm{d}x=\left[\frac{x^{2a+1}}{2a+1}\right]_0^1=\frac{1}{2a+1}$	M1 A1	Finds $\int_0^1 y^2\,\mathrm{d}x$.
	$(\bar{x},\bar{y})=\left(\frac{a+1}{a+2},\frac{a+1}{2(2a+1)}\right)$	A1	Both coordinates correct.
		6	

Question	Answer	Marks	Guidance
2	$\frac{dy}{dx}=\frac{a}{ax+1}=(-1)^0\frac{0!a^1}{(ax+1)^1}$ so true for $n=1$.	**M1 A1**	Proves base case.
	Assume that $\frac{d^k y}{dx^k}=(-1)^{k-1}\frac{(k-1)!a^k}{(ax+1)^k}$ for some positive integer k.	**B1**	States inductive hypothesis.
	Then $\frac{d^{k+1}y}{dx^{k+1}}=-ka(-1)^{k-1}\frac{(k-1)!a^k}{(ax+1)^{k+1}}=(-1)^k\frac{k!a^{k+1}}{(ax+1)^{k+1}}$ so true for $n=k+1$.	**M1 A1**	Differentiates k th derivative.
	By induction, true for every positive integer n.	**A1**	States conclusion.
		6	

Question	Answer	Marks	Guidance
3(i)	$I_{n+2} = \left[\frac{x^{-n-1}}{-n-1}\sin \pi x\right]_{\frac{1}{2}}^{1} - \pi\int_{\frac{1}{2}}^{1}\frac{x^{-n-1}}{-n-1}\cos \pi x\,\mathrm{d}x$	M1 A1	Integrates by parts.
	$= \frac{2^{n+1}}{n+1} + \frac{\pi}{n+1}\left(\left[\frac{x^{-n}}{-n}\cos \pi x\right]_{\frac{1}{2}}^{1} + \pi\int_{\frac{1}{2}}^{1}\frac{x^{-n}}{-n}\sin \pi x\,\mathrm{d}x\right)$	M1	Integrates by parts again.
	$= \frac{2^{n+1}}{n+1} + \frac{\pi}{n+1}\left(\frac{1}{n} - \frac{\pi}{n}I_n\right)$ $\Rightarrow (n+1)I_{n+2} = 2^{n+1} + \pi\left(\frac{1}{n} - \frac{\pi}{n}I_n\right)$	M1	Uses I_n.
	$\Rightarrow n(n+1)I_{n+2} = 2^{n+1}n + \pi - \pi^2 I_n$	A1	AG
		5	
3(ii)	$2I_3 = 4 + \pi - \pi^2 I_1$ $12I_5 = 48 + \pi - \frac{\pi^2}{2}\left(4 + \pi - \pi^2 I_1\right)$	M1	Substitutes I_3 into reduction formula.
	$\Rightarrow I_5 = 4 + \frac{1}{24}\left(2\pi - 4\pi^2 - \pi^3 + \pi^4 I_1\right)$	A1	AEF, must be exact with fractions simplified.
		2	

Question	Answer	Marks	Guidance
4(i)	$x^2+1=(ax+b)(2x+1)+c$	M1	Uses that $2x+1$ is the quotient.
	$\Rightarrow a=\frac{1}{2},\ b=-\frac{1}{4}$	A1 A1	
		3	
4(ii)	$x=\frac{1}{2}$	B1 FT	
		1	

Question	Answer	Marks	Guidance
4(iii)		B1	Intersection (0,-4) given and asymptotes drawn.
		B1	Left branch correct.
		B1 FT	Right branch correct. Deduct at most one mark for poor forms at infinity.
		3	

Question	Answer	Marks	Guidance
5(i)	$\sum_{r=1}^{N}(5r+1)(5r+6)=25\sum_{r=1}^{N}r^2+35\sum_{r=1}^{N}r+6N$	M1	Expands.
	$25\left(\frac{1}{6}N(N+1)(2N+1)\right)+35\left(\frac{1}{2}N(N+1)\right)+6N$	M1	Substitutes formulae for $\sum r$ and $\sum r^2$.
	$=N\left(\frac{25}{6}(2N^2+3N+1)+\frac{35}{2}N+\frac{35}{2}+6\right)=\frac{1}{3}N(25N^2+90N+83)$	A1	Simplifies to the given answer (AG).
		3	
5(ii)	$\frac{1}{(5r+1)(5r+6)}=\frac{1}{5}\left(\frac{1}{5r+1}-\frac{1}{5r+6}\right)$	M1 A1	Finds partial fractions.
	$T_N=\frac{1}{5}\left(\frac{1}{6}-\frac{1}{11}+\frac{1}{11}-\frac{1}{16}+\cdots+\frac{1}{5N+1}-\frac{1}{5N+6}\right)$	M1	Expresses terms as differences.
	$\frac{1}{5}\left(\frac{1}{6}-\frac{1}{5N+6}\right)=\frac{1}{30}-\frac{1}{5(5N+6)}$	A1	At least 3 terms including last.
		4	
5(iii)	$\frac{S_N}{N^3}T_N\to\frac{25}{3}\times\frac{1}{30}=\frac{5}{18}$	M1 A1	Divides S_N by N^3 and takes limits as $N\to\infty$
		2	

Question	Answer	Marks	Guidance
6(i)	$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$	B1	
	$\overrightarrow{OC} \times \overrightarrow{AB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 1 & 2 & 7 \end{vmatrix} = \begin{pmatrix} -9 \\ -6 \\ 3 \end{pmatrix} = t\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$	M1 A1	Finds direction of common perpendicular.
	$\dfrac{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}}{\sqrt{3^2+2^2+1^2}} = \dfrac{1}{\sqrt{14}} = 0.267$	M1 A1	Uses formula for shortest distance.
		5	
6(ii)	$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 3 & 2 & -1 \end{vmatrix} = t\begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix}$	M1 A1	Finds normal to plane.
	$-(0)+4(0)+5(0)=0$	M1	Uses point on plane.
	$-x+4y+5z=0$	A1	AEF
		4	

Question	Answer	Marks	Guidance
7(i)	$\sqrt{-7y}(-7y)+2(-7y)+\sqrt{-7y}+7=0$ $\Rightarrow \sqrt{-7y}(-7y+1)=14y-7 \Rightarrow -7y(-7y+1)^2=(14y-7)^2$	M1	Uses given substitution and eliminates radical.
	$\Rightarrow 49y^3+14y^2-27y+7=0$	A1	AG
	$y=\dfrac{x^2}{-7}=\dfrac{x^2}{\alpha\beta\gamma}$	M1	Uses $\alpha\beta\gamma=-7$.
	So roots are $\dfrac{\alpha^2}{\alpha\beta\gamma}=\dfrac{\alpha}{\beta\gamma}, \dfrac{\beta^2}{\alpha\beta\gamma}=\dfrac{\beta}{\alpha\gamma}, \dfrac{\gamma^2}{\alpha\beta\gamma}=\dfrac{\gamma}{\alpha\beta}$	A1	AG
		4	
7(ii)	$\dfrac{\alpha}{\beta\gamma}+\dfrac{\beta}{\alpha\gamma}+\dfrac{\gamma}{\alpha\beta}=-\dfrac{2}{7}, \dfrac{1}{\gamma^2}+\dfrac{1}{\beta^2}+\dfrac{1}{\alpha^2}=-\dfrac{27}{49}$	B1	States sum of roots and $\alpha'\beta'+\alpha'\gamma'+\beta'\gamma'$.
	$\dfrac{\alpha^2}{\beta^2\gamma^2}+\dfrac{\beta^2}{\alpha^2\gamma^2}+\dfrac{\gamma^2}{\alpha^2\beta^2}=\left(-\dfrac{2}{7}\right)^2-2\left(-\dfrac{27}{49}\right)=\dfrac{58}{49}$	M1 A1	Uses $\alpha'^2+\beta'^2+\gamma'^2=(\alpha'+\beta'+\gamma')^2-2(\alpha'\beta'+\alpha'\gamma'+\beta'\gamma')$ AG
		3	
7(iii)	$49\left(\dfrac{\alpha^3}{\beta^3\gamma^3}+\dfrac{\beta^3}{\alpha^3\gamma^3}+\dfrac{\gamma^3}{\alpha^3\beta^3}\right)=-14\left(\dfrac{58}{49}\right)+27\left(-\dfrac{2}{7}\right)-21$	M1	Uses $49\alpha'^3=-14\alpha'^2+27\alpha'-7$.
	$\Rightarrow \dfrac{\alpha^3}{\beta^3\gamma^3}+\dfrac{\beta^3}{\alpha^3\gamma^3}+\dfrac{\gamma^3}{\alpha^3\beta^3}=-\dfrac{317}{343}$	A1	
		2	

Question	Answer	Marks	Guidance
8(i)	Eigenvalues of (upper diagonal matrix) $\mathbf{A}$ are $2, m$ and 1. (Or from characteristic equation: $(\lambda-2)(\lambda-m)(\lambda-1)=0$)	B1	
	$\lambda=2:\ \mathbf{e}_1=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & m-2 & 1\\ 0 & 0 & -1\end{vmatrix}=\begin{pmatrix}2-m\\0\\0\end{pmatrix}=t\begin{pmatrix}1\\0\\0\end{pmatrix}$	M1 A1	Uses vector product (or equations) to find corresponding eigenvectors.
	$\lambda=m:\ \mathbf{e}_2=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2-m & m & 1\\ 0 & 0 & 7\end{vmatrix}=\begin{pmatrix}7m\\7(m-2)\\0\end{pmatrix}=t\begin{pmatrix}m\\m-2\\0\end{pmatrix}$	A1	
	$\lambda=1:\ \mathbf{e}_3=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & m & 1\\ 0 & m-1 & 7\end{vmatrix}=\begin{pmatrix}6m+1\\-7\\m-1\end{pmatrix}=t\begin{pmatrix}6m+1\\-7\\m-1\end{pmatrix}$	A1	
	Thus $\mathbf{P}=\begin{pmatrix}1 & m & 6m+1\\ 0 & m-2 & -7\\ 0 & 0 & m-1\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}2 & 0 & 0\\ 0 & m & 0\\ 0 & 0 & 1\end{pmatrix}$	M1 A1 FT	Or correctly matched permutations of columns. No follow through on two or more zero eigenvectors.
		7	

Question	Answer	Marks	Guidance
8(ii)	$\mathbf{M}^7\mathbf{P} = \mathbf{P}\mathbf{D}^7\mathbf{P}^{-1}\mathbf{P} = \mathbf{P}\mathbf{D}^7 = \begin{pmatrix} 1 & m & 6m+1 \\ 0 & m-2 & -7 \\ 0 & 0 & m-1 \end{pmatrix}\begin{pmatrix} 2^7 & 0 & 0 \\ 0 & m^7 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	M1 A1 FT	Applies $\mathbf{M}^7 = \mathbf{P}\mathbf{D}^7\mathbf{P}^{-1}$.
	$= \begin{pmatrix} 2^7 & m^8 & 6m+1 \\ 0 & m^8 - 2m^7 & -7 \\ 0 & 0 & m-1 \end{pmatrix}$	A1	Order of columns might be swapped depending on **P**.
		3	

Question	Answer	Marks	Guidance
9(i)	Write $c = \cos\theta$, $s = \sin\theta$. $\cos 6\theta + i\sin 6\theta = (c + is)^6$	M1	Uses binomial theorem.
	$\Rightarrow \cos 6\theta = c^6 - 15c^4s^2 + 15c^2s^4 - s^6$	A1	
	$c^6 - 15c^4s^2 + 15c^2s^4 - s^6 = c^6 - 15c^4(1-c^2) + 15c^2(1-c^2)^2 - (1-c^2)^3$	M1	Uses $c^2 = 1 - s^2$.
	$= c^6 - 15c^4(1-c^2) + 15c^2(1 - 2c^2 + c^4) - (1 - 3c^2 + 3c^4 - c^6)$	A1	
	$= 32c^6 - 48c^4 + 18c^2 - 1$	M1	Divides numerator and denominator by c^6.
	$\Rightarrow \sec 6\theta = \dfrac{1}{32c^6 - 48c^4 + 18c^2 - 1} = \dfrac{\sec^6\theta}{32 - 48\sec^2\theta + 18\sec^4\theta - \sec^6\theta}$	A1	AG
		6	

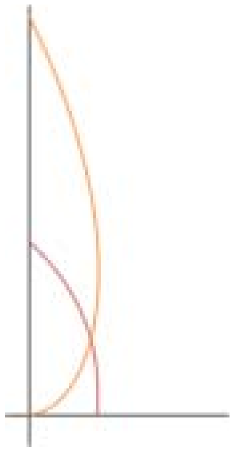
Question	Answer	Marks	Guidance
9(ii)	$x^6 = 2\left(32 - 48x^2 + 18x^4 - x^6\right) \Rightarrow \dfrac{x^6}{32 - 48x^2 + 18x^4 - x^6} = 2$	M1 A1	Relates with equation in part (i).
	$\sec 6\theta = 2 \Rightarrow \cos 6\theta = \dfrac{1}{2}$	M1	Solves $\cos 6\theta = \dfrac{1}{2}$.
	$x = \sec\dfrac{\pi}{18}$	A1	Gives one correct solution.
	$x = \sec q\pi$, $q = \dfrac{5}{18}, \dfrac{7}{18}, \dfrac{11}{18}, \dfrac{13}{18}, \dfrac{17}{18}$	A1	Gives five other solutions. Allow different values of q as long as all six solutions are found.
		5	

Question	Answer	Marks	Guidance
10(i)	$\begin{pmatrix}1&5&1\\1&-2&-2\\2&3&\theta\end{pmatrix}\rightarrow\begin{pmatrix}1&5&1\\0&-7&-3\\0&0&\theta+1\end{pmatrix}$	M1 A1	Reduces to echelon form. At least one row operation for M1.
	$r(\mathbf{A})=3$ if $\theta\neq-1$	A1	
	$r(\mathbf{A})=2$ if $\theta=-1$	B1	
		4	
10(ii)	$\begin{aligned}x+5y+z&=-1\\-7y-3z&=1\\(\theta+1)z&=(\theta+1)\end{aligned}$	M1	Uses reduced form of augmented matrix or eliminates variables from scratch.
	$z=1,\ y=-\frac{4}{7},\ x=\frac{6}{7}$	A1 A1	One correct. All three correct.
		3	
10(iii)	$\begin{aligned}x+5y+z&=-1\\-7y-3z&=1\\(\theta+1)z&=(\theta+1)\end{aligned}$ $z=t$	M1	Uses parameter.
	$y=-\frac{3t+1}{7},\ x=\frac{8t-2}{7}$	A1 A1	
		3	

Question	Answer	Marks	Guidance
10(iv)	$x+5y+\quad z=-1$, $-7y\quad -3z=1$, $(\theta+1)z=\phi+1$	M1	Uses reduced form of augmented matrix or eliminates variables from scratch
	$\theta=-1\Rightarrow\phi=-1$ so no solution (inconsistent).	A1	
		2	

Question	Answer	Marks	Guidance
11E(i)	$w=\cos y\Rightarrow\frac{\mathrm{d}w}{\mathrm{d}x}=-\sin y\frac{\mathrm{d}y}{\mathrm{d}x}$	B1	
	$\frac{\mathrm{d}^2w}{\mathrm{d}x^2}=-\sin y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\cos y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$	B1	
	$\frac{\mathrm{d}^2w}{\mathrm{d}x^2}+2\frac{\mathrm{d}w}{\mathrm{d}x}+w=-\sin y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\cos y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2-2\sin y\frac{\mathrm{d}y}{\mathrm{d}x}+\cos y$	M1	Uses substitution to obtain w - x equation, AG.
	$=-\cos y\left(\mathrm{e}^{-2x}\sec y\right)=-\mathrm{e}^{-2x}$	A1	
		4	

Question	Answer	Marks	Guidance
11E(ii)	$m^2+2m+1=0 \Rightarrow m=-1$	M1	Finds CF.
	CF: $w=(Ax+B)e^{-x}$	A1	
	PI: $w=ke^{-2x} \Rightarrow w'=-2ke^{-2x} \Rightarrow w''=4ke^{-2x}$	M1	Forms PI and differentiates.
	$4k-4k+k=-1 \Rightarrow k=-1$	A1	
	$w=(Ax+B)e^{-x}-e^{-2x}$	A1	States general solution.
	$x=0$, $y=\frac{1}{3}\pi$, $w=\frac{1}{2} \Rightarrow B=\frac{3}{2}$	B1	Uses initial conditions to find constants.
	$w'=-(Ax+B)e^{-x}+Ae^{-x}+2e^{-2x}$	M1	Differentiates general solution.
	$x=0$, $y=\frac{1}{3}\pi$, $y'=\frac{\sqrt{3}}{3}$, $w'=-\frac{1}{2} \Rightarrow -\frac{1}{2}=-\frac{3}{2}+A+2 \Rightarrow A=-1$	M1 A1	Substitutes initial conditions.
	$y=\cos^{-1}\left(\left(\frac{3}{2}-x\right)e^{-x}-e^{-2x}\right)$	A1	States particular solution for y in terms of x.
		10	

Question	Answer	Marks	Guidance
11O(i)	$e^{2\alpha} - e^{-2\alpha} = 2(e^{\alpha} + e^{-\alpha}) \Rightarrow e^{\alpha} - e^{-\alpha} = 2$	M1	Sets equations equal and divides by $e^{\alpha} + e^{-\alpha}$.
	$e^{2\alpha} - 2e^{\alpha} - 1 = 0 \Rightarrow e^{\alpha} = 1 + \sqrt{2}$	M1 A1	Forms quadratic in e^{α}, AG.
	$\alpha = \ln(1 + \sqrt{2})$	A1	Must be exact.
	$r = 2(1 + \sqrt{2} + \sqrt{2} - 1) = 4\sqrt{2}$	M1 A1	Substitutes to find r.
		6	
11O(ii)		B1	C_1 has correct shape.
		B1	C_2 has correct shape.
		B1	Intersection points positioned correctly.
		3	

Question	Answer	Marks	Guidance
11O(iii)	$2\int_0^{\ln(1+\sqrt{2})}\left(e^{\theta}+e^{-\theta}\right)^2\ d\theta-\frac{1}{2}\int_0^{\ln(1+\sqrt{2})}\left(e^{2\theta}-e^{-2\theta}\right)^2\ d\theta$ $=\int_0^{\ln(1+\sqrt{2})}5+2e^{2\theta}+2e^{-2\theta}-\frac{1}{2}e^{4\theta}-\frac{1}{2}e^{-4\theta}\ d\theta$	M1 A1	Uses $\frac{1}{2}\int r^2\ d\theta$ to formulate correct area.
	$=\left[5\theta+e^{2\theta}-e^{-2\theta}-\frac{1}{8}e^{4\theta}+\frac{1}{8}e^{-4\theta}\right]_0^{\ln(1+\sqrt{2})}$	M1 A1	Expands and integrates.
	$=5\ln(1+\sqrt{2})+\left(1+\sqrt{2}\right)^2-\left(1+\sqrt{2}\right)^{-2}-\frac{1}{8}\left(\left(1+\sqrt{2}\right)^4-\left(1+\sqrt{2}\right)^{-4}\right)=5.82$	A1	
		5	

Cambridge International A Level

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Cambridge International Advanced Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

5631547597

FURTHER MATHEMATICS **9231/11**

Paper 1 **May/June 2019**

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF10)

READ THESE INSTRUCTIONS FIRST

Write your centre number, candidate number and name in the spaces at the top of this page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.
DO **NOT** WRITE IN ANY BARCODES.

Answer **all** the questions in the space provided. If additional space is required, you should use the lined page at the end of this booklet. The question number(s) must be clearly shown.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **26** printed pages and **2** blank pages.

Cambridge Assessment International Education

1 A curve C has equation $\cos y = x$, for $-\pi < x < \pi$.

(i) Use implicit differentiation to show that

$$\frac{d^2y}{dx^2} = -\cot y\left(\frac{dy}{dx}\right)^2.$$ [4]

(ii) Hence find the exact value of $\frac{d^2y}{dx^2}$ at the point $\left(\frac{1}{2}, \frac{1}{3}\pi\right)$ on C. [2]

2 Let $u_n = \dfrac{4\sin(n-\frac{1}{2})\sin\frac{1}{2}}{\cos(2n-1)+\cos 1}$.

(i) Using the formulae for $\cos P \pm \cos Q$ given in the List of Formulae MF10, show that

$$u_n = \frac{1}{\cos n} - \frac{1}{\cos(n-1)}.$$ [2]

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(ii) Use the method of differences to find $\sum_{n=1}^{N} u_n$. [2]

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(iii) Explain why the infinite series $u_1 + u_2 + u_3 + \ldots$ does not converge. [1]

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3 The lines l_1 and l_2 have equations $\mathbf{r} = 6\mathbf{i} + 2\mathbf{j} + 7\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j})$ and $\mathbf{r} = 4\mathbf{i} + 4\mathbf{j} + \mu(-6\mathbf{j} + \mathbf{k})$ respectively. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. Find the position vectors of P and Q. [8]

4 It is given that, for $n \geqslant 0$,

$$I_n = \int_0^1 x^n e^{x^3}\, dx.$$

(i) Show that $I_2 = \frac{1}{3}(e - 1)$. [2]

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(ii) Show that, for $n \geqslant 3$,

$$3I_n = e - (n-2)I_{n-3}.$$ [3]

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(iii) Hence find the exact value of I_8. [3]

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5 A curve C is defined parametrically by

$$x = \frac{2}{e^t + e^{-t}} \quad \text{and} \quad y = \frac{e^t - e^{-t}}{e^t + e^{-t}},$$

for $0 \leqslant t \leqslant 1$. The area of the surface generated when C is rotated through 2π radians about the x-axis is denoted by S.

(i) Show that $S = 4\pi \int_0^1 \frac{e^t - e^{-t}}{(e^t + e^{-t})^2}\, dt$. [5]

(ii) Using the substitution $u = e^t + e^{-t}$, or otherwise, find S in terms of π and e. [3]

6 The equation

$$x^3 - x + 1 = 0$$

has roots α, β, γ.

(i) Use the relation $x = y^{\frac{1}{3}}$ to show that the equation

$$y^3 + 3y^2 + 2y + 1 = 0$$

has roots α^3, β^3, γ^3. Hence write down the value of $\alpha^3 + \beta^3 + \gamma^3$. [3]

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Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(ii) Find the value of S_{-3}. [2]

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(iii) Show that $S_6 = 5$ and find the value of S_9. [4]

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7 Find the particular solution of the differential equation

$$10\frac{d^2x}{dt^2} + 3\frac{dx}{dt} - x = t + 2,$$

given that when $t = 0$, $x = 0$ and $\frac{dx}{dt} = 0$. [10]

8 **(i)** Prove by mathematical induction that, for $z \neq 1$ and all positive integers n,

$$1 + z + z^2 + \ldots + z^{n-1} = \frac{z^n - 1}{z - 1}.$$ [5]

(ii) By letting $z = \frac{1}{2}(\cos\theta + \mathrm{i}\sin\theta)$, use de Moivre's theorem to deduce that

$$\sum_{m=1}^{\infty} \left(\tfrac{1}{2}\right)^m \sin m\theta = \frac{2\sin\theta}{5 - 4\cos\theta}.$$ [5]

9 It is given that **e** is an eigenvector of the matrix **A**, with corresponding eigenvalue λ.

(i) Show that **e** is an eigenvector of $\mathbf{A}^2$, with corresponding eigenvalue λ^2. [2]

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The matrices **A** and **B** are given by

$$\mathbf{A} = \begin{pmatrix} n & 1 & 3 \\ 0 & 2n & 0 \\ 0 & 0 & 3n \end{pmatrix} \quad \text{and} \quad \mathbf{B} = (\mathbf{A} + n\mathbf{I})^2,$$

where **I** is the 3×3 identity matrix and n is a non-zero integer.

(ii) Find, in terms of n, a non-singular matrix **P** and a diagonal matrix **D** such that $\mathbf{B} = \mathbf{PDP}^{-1}$. [8]

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10 The curves C_1 and C_2 have equations

$$y = \frac{ax}{x+5} \quad \text{and} \quad y = \frac{x^2 + (a+10)x + 5a + 26}{x+5}$$

respectively, where a is a constant and $a > 2$.

(i) Find the equations of the asymptotes of C_1. [2]

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(ii) Find the equation of the oblique asymptote of C_2. [2]

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(iii) Show that C_1 and C_2 do not intersect. [2]

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(iv) Find the coordinates of the stationary points of C_2. [3]

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(v) Sketch C_1 and C_2 on a single diagram. [You do not need to calculate the coordinates of any points where C_2 crosses the axes.] [3]

11 Answer only **one** of the following two alternatives.

EITHER

The curve C_1 has polar equation $r^2 = 2\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(i) The point on C_1 furthest from the line $\theta = \frac{1}{2}\pi$ is denoted by P. Show that, at P,

$$2\theta \tan\theta = 1$$

and verify that this equation has a root between 0.6 and 0.7. [5]

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The curve C_2 has polar equation $r^2 = \theta \sec^2 \theta$, for $0 \leqslant \theta < \frac{1}{2}\pi$. The curves C_1 and C_2 intersect at the pole, denoted by O, and at another point Q.

(ii) Find the exact value of θ at Q. [2]

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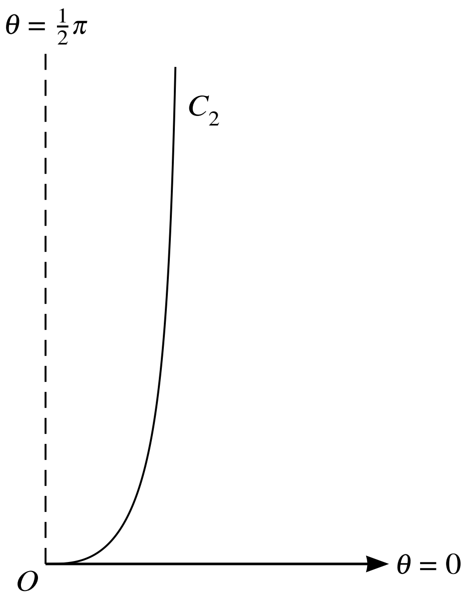
(iii) The diagram below shows the curve C_2. Sketch C_1 on this diagram. [2]



(iv) Find, in exact form, the area of the region OPQ enclosed by C_1 and C_2. [5]

OR

The linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix

$$\mathbf{M} = \begin{pmatrix} -1 & 2 & 3 & 4 \\ 1 & 0 & 1 & -1 \\ 1 & -2 & -3 & a \\ 1 & 2 & 5 & 2 \end{pmatrix}.$$

(i) For $a \neq -4$, the range space of T is denoted by V.

(a) Find the dimension of V and show that

$$\begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 0 \\ -2 \\ 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 4 \\ -1 \\ a \\ 2 \end{pmatrix}$$

form a basis for V. [5]

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(b) Show that if $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ belongs to V then $x + 2y = t$. [4]

(ii) For $a = -4$, find the general solution of

$$\mathbf{Mx} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$ [5]

Additional Page

If you use the following lined page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.

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FURTHER MATHEMATICS **9231/11**

Paper 1 **May/June 2019**

MARK SCHEME

Maximum Mark: 100

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

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Cambridge International is publishing the mark schemes for the May/June 2019 series for most Cambridge IGCSE™, Cambridge International A and AS Level and Cambridge Pre-U components, and some Cambridge O Level components.

Question	Answer	Marks	Guidance
1(i)	$-\sin y\frac{\mathrm{d}y}{\mathrm{d}x}=1\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=-(\sin y)^{-1}$	**M1 A1**	Differentiates implicitly once.
	$\frac{\mathrm{d}^2y}{\mathrm{d}^2x}=(\sin y)^{-2}\cos y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=-\cot y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$	**M1 A1**	Differentiates again, AG.
		4	
1(ii)	$\frac{\mathrm{d}y}{\mathrm{d}x}=-\left(\sin\frac{\pi}{3}\right)^{-1}=-\frac{2}{\sqrt{3}}$	**B1**	
	$\frac{\mathrm{d}^2y}{\mathrm{d}^2x}=-\frac{1}{\sqrt{3}}\left(-\frac{2}{\sqrt{3}}\right)^2=-\frac{4}{3\sqrt{3}}=-\frac{4}{9}\sqrt{3}$	**B1**	AEF, must be exact.
		2	

Question	Answer	Marks	Guidance
2(i)	$\frac{4\sin\left(n-\frac{1}{2}\right)\sin\frac{1}{2}}{\cos(2n-1)+\cos 1}=\frac{2(\cos(n-1)-\cos n)}{2\cos n\cos(n-1)}$	**M1**	Uses formulae for $\cos P \pm \cos Q$.
	$\frac{1}{\cos n}-\frac{1}{\cos(n-1)}$	**A1**	AG
		2	
2(ii)	$\sum_{n=1}^{N}\frac{4\sin\left(n-\frac{1}{2}\right)\sin\frac{1}{2}}{\cos(2n-1)+\cos 1}=\sum_{n=1}^{N}\frac{1}{\cos n}-\frac{1}{\cos(n-1)}$ $=\frac{1}{\cos 1}-\frac{1}{1}+\frac{1}{\cos 2}-\frac{1}{\cos 1}+\ldots+\frac{1}{\cos N}-\frac{1}{\cos(N-1)}$	**M1**	Applies (i), shows enough terms and cancelation.
	$-1+\frac{1}{\cos N}$	**A1**	
		2	
2(iii)	$\cos N$ oscillates as $N\to\infty$ so $u_1+u_2+u_3+\ldots$ does not converge.	**B1**	States "oscillates" or refers to diverging values of $\cos N$.
		1	

Question	Answer	Marks	Guidance
3	$\overrightarrow{OP}=\begin{pmatrix}6+\lambda\\2+\lambda\\7\end{pmatrix},\overrightarrow{OQ}=\begin{pmatrix}4\\4-6\mu\\\mu\end{pmatrix}\Rightarrow\overrightarrow{PQ}=\begin{pmatrix}-2-\lambda\\2-\lambda-6\mu\\-7+\mu\end{pmatrix}$	M1 A1	Finds $\overrightarrow{PQ}$.
	$\begin{pmatrix}-2-\lambda\\2-\lambda-6\mu\\-7+\mu\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix}=0$ **Or** $\begin{pmatrix}-2-\lambda\\2-\lambda-6\mu\\-7+\mu\end{pmatrix}=k\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&0\\0&-6&1\end{vmatrix}=k\begin{pmatrix}1\\-1\\-6\end{pmatrix}$	M1	Uses that dot product of $\overrightarrow{PQ}$ with line directions is 0. Or, alternatively, $\overrightarrow{PQ}$ is multiple of common perpendicular.
	$-2\lambda-6\mu=0$	A1	Deduces one equation. CWO.
	$\begin{pmatrix}-2-\lambda\\2-\lambda-6\mu\\-7+\mu\end{pmatrix}\begin{pmatrix}0\\-6\\1\end{pmatrix}=0\Rightarrow 6\lambda+37\mu=19$	A1	Deduces second equation. CWO.
	$\lambda=-3,\ \mu=1$	M1 A1	Solves simultaneous equations.
	$\overrightarrow{OP}=\begin{pmatrix}3\\-1\\7\end{pmatrix},\overrightarrow{OQ}=\begin{pmatrix}4\\-2\\1\end{pmatrix}$	A1	States $\overrightarrow{OP}$ and $\overrightarrow{OQ}$.
		8	

Question	Answer	Marks	Guidance
4(i)	$\int_0^1 x^2 e^{x^3}\ \mathrm{d}x = \frac{1}{3}\left[e^{x^3}\right]_0^1 = \frac{1}{3}(e-1)$	**M1 A1**	Must show working, AG.
		2	
4(ii)	$I_n = \int_0^1 x^{n-2}x^2 e^{x^3}\ \mathrm{d}x = \left[\frac{1}{3}x^{n-2}e^{x^3}\right]_0^1 - \frac{n-2}{3}\int_0^1 x^{n-3}e^{x^3}\ \mathrm{d}x$	**M1 A1**	Integrates by parts.
	$\frac{e}{3} - \frac{n-2}{3}I_{n-3} \Rightarrow 3I_n = e-(n-2)I_{n-3}$	**A1**	AG
		3	
4(iii)	$I_5 = \frac{1}{3}(e-3I_2) = \frac{1}{3}(e-e+1) = \frac{1}{3}$	**M1 A1**	Applies reduction formula once and uses **(i)**.
	$I_8 = \frac{1}{3}(e-6I_5) = \frac{1}{3}(e-2)$.	**A1**	Must be exact.
		3	

Question	Answer	Marks	Guidance
5(i)	$\frac{dx}{dt} = \frac{-2\left(e^t - e^{-t}\right)}{\left(e^t + e^{-t}\right)^2}$	B1	
	$\frac{dy}{dt} = \frac{\left(e^t + e^{-t}\right)^2 - \left(e^t - e^{-t}\right)^2}{\left(e^t + e^{-t}\right)^2} = \frac{\left(2e^t\right)\left(2e^{-t}\right)}{\left(e^t + e^{-t}\right)^2} = \frac{4}{\left(e^t + e^{-t}\right)^2}$	B1	Differentiates and simplifies. Accept $1 - \left(\frac{e^t - e^{-t}}{e^t + e^{-t}}\right)^2$.
	$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{4\left(e^t - e^{-t}\right)^2 + 16}{\left(e^t + e^{-t}\right)^4} = \frac{4\left(e^t + e^{-t}\right)^2}{\left(e^t + e^{-t}\right)^4}$	M1 A1	Attempt at writing $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$ as a square.
	$S = 2\pi\int_0^1 \left(\frac{e^t - e^{-t}}{e^t + e^{-t}}\right)\left(\frac{2}{e^t + e^{-t}}\right)dt = 4\pi\int_0^1 \frac{e^t - e^{-t}}{\left(e^t + e^{-t}\right)^2}dt$	A1	Uses correct formula, simplifies to AG.
		5	
5(ii)	$S = 4\pi\int_2^{e+e^{-1}} u^{-2}\,du = 4\pi\left[-u^{-1}\right]_2^{e+e^{-1}}$	M1 A1	Applies given substitution.
	$= 4\pi\left(\frac{1}{2} - \frac{1}{e + e^{-1}}\right)$.	A1	AEF, must be exact.
		3	

Question	Answer	Marks	Guidance
6(i)	$y=(y+1)^3$	M1	Obtains an equation in y not involving radicals.
	$y=y^3+3y^2+3y+1 \Rightarrow y^3+3y^2+2y+1=0$	A1	AG
	$S_3=-3$	B1	
		3	
6(ii)	$S_{-3}=\dfrac{\alpha^3\beta^3+\beta^3\gamma^3+\alpha^3\gamma^3}{\alpha^3\beta^3\gamma^3}=\dfrac{2}{-1}=-2$	M1 A1	
		2	
6(iii)	$S_6=(-3)^2-2(2)=5$	M1 A1	Uses $(\Sigma\alpha)^2=\Sigma\alpha^2+2\sum_{\alpha\neq\beta}\alpha\beta$. AG.
	$S_9=-3S_6-2S_3-3=-3(5)-2(-3)-3=-12$	M1 A1	Sums $y^3-3y^2+2y-1=0$ for $y=\alpha^3,\beta^3,\gamma^3$.
		4	

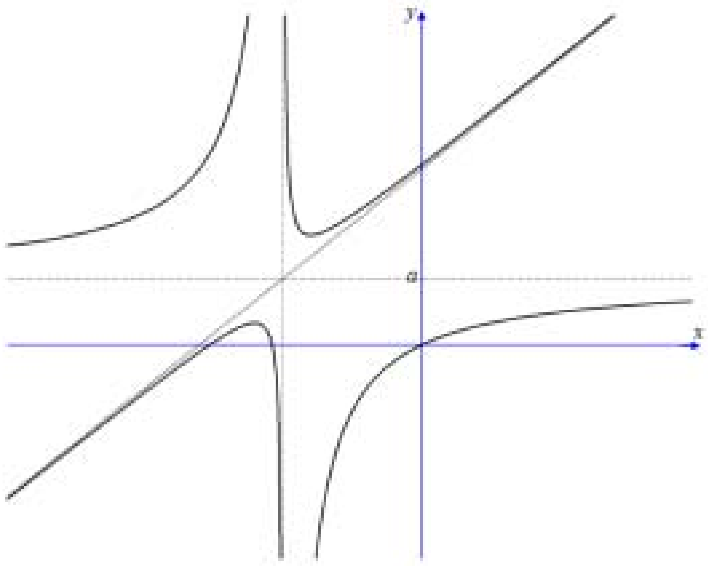
Question	Answer	Marks	Guidance
7	$10u^2+3u-1=0 \Rightarrow (2u+1)(5u-1)=0$	M1	Axillary equation
	CF: $x = Ae^{-\frac{1}{2}t}+Be^{\frac{1}{5}t}$	A1	
	PI: $x=p+qt \Rightarrow \dot{x}=q \Rightarrow \ddot{x}=0$	M1	Forms PI and differentiates.
	$3q-p-qt=t+2 \Rightarrow q=-1, p=-5.$	M1 A1	Substitutes.
	GS: $x = Ae^{-\frac{1}{2}t}+Be^{\frac{1}{5}t}-t-5$	A1	States general solution.
	$\dot{x}=-\frac{1}{2}Ae^{-\frac{1}{2}t}+\frac{1}{5}Be^{\frac{1}{5}t}-1$	M1	Differentiates.
	$A+B=5$ $-\frac{1}{2}A+\frac{1}{5}B=1$	M1	Forms simultaneous equations.
	$A=0,\ B=5$	A1	
	$x=5e^{\frac{1}{5}t}-t-5$	A1	States PS.
		10	

Question	Answer	Marks	Guidance
8(i)	$1=\dfrac{z^1-1}{z-1}$ So true when $n=1$.	**B1**	Shows base case.
	Assume that $1+z+\ldots+z^{k-1}=\dfrac{z^k-1}{z-1}$	**B1**	States inductive hypothesis.
	Then $1+z+\ldots+z^{k-1}+z^k=\dfrac{z^k-1}{z-1}+z^k=\dfrac{z^k-1+z^k(z-1)}{z-1}=\dfrac{z^{k+1}-1}{z-1}$, so true when $n=k+1$	**M1 A1**	Combines fractions.
	$H_k \to H_{k+1}$ Hence, by induction, true for all positive integers.	**A1**	States conclusion.
		5	

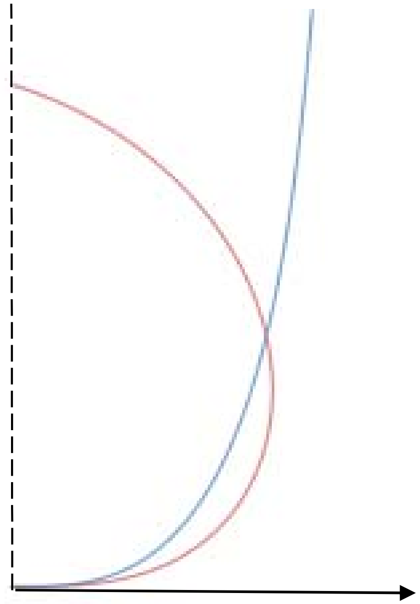
Question	Answer	Marks	Guidance
8(ii)	Since $\lvert z\rvert<1$, $\sum_{m=0}^{\infty} z^m = \frac{-1}{z-1}$	**B1**	States $\lvert z\rvert<1$ and uses formula for sum to infinity of geometric progression.
	$\sum_{m=1}^{\infty} 2^{-m}\sin m\theta = \mathrm{Im}\left(\sum_{m=0}^{\infty} z^m\right) = \mathrm{Im}\left(\frac{-1}{\frac{1}{2}\cos\theta + \mathrm{i}\frac{1}{2}\sin\theta - 1}\right)$	**M1 A1**	Uses de Moivre's theorem.
	$\mathrm{Im}\left(\frac{-\left(\frac{1}{2}\cos\theta - 1 - \mathrm{i}\frac{1}{2}\sin\theta\right)}{\frac{1}{4}\cos^2\theta - \cos\theta + 1 + \frac{1}{4}\sin^2\theta}\right)$	**M1**	Multiply numerator and denominator by conjugate.
	$\frac{\frac{1}{2}\sin\theta}{\frac{5}{4}-\cos\theta} = \frac{2\sin\theta}{5-4\cos\theta}$	**A1**	States imaginary part, AG.
		5	

Question	Answer	Marks	Guidance
9(i)	$\mathbf{A}^2\mathbf{e} = \mathbf{A}(\mathbf{Ae}) = \lambda\mathbf{Ae} = \lambda^2\mathbf{e}$	M1 A1	AG
		2	
9(ii)	Eigenvalues of $\mathbf{A}$ are $n, 2n$ and $3n$.	B1	
	$\lambda = n$: $\mathbf{e}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 3 \\ 0 & n & 0 \end{vmatrix} = \begin{pmatrix} -3n \\ 0 \\ 0 \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	M1 A1	Uses vector product (or equations) to find corresponding eigenvectors.
	$\lambda = 2n$: $\mathbf{e}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -n & 1 & 3 \\ 0 & 0 & n \end{vmatrix} = \begin{pmatrix} n \\ n^2 \\ 0 \end{pmatrix} = t\begin{pmatrix} 1 \\ n \\ 0 \end{pmatrix}$	A1	
	$\lambda = 3n$: $\mathbf{e}_3 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2n & 1 & 3 \\ 0 & -n & 0 \end{vmatrix} = \begin{pmatrix} 3n \\ 0 \\ 2n^2 \end{pmatrix} = t\begin{pmatrix} 3 \\ 0 \\ 2n \end{pmatrix}$	A1	
	Eigenvalues of $\mathbf{A} + n\mathbf{I}$ are $2n, 3n$ and $4n$	B1	
	Thus $\mathbf{P} = \begin{pmatrix} 1 & 1 & 3 \\ 0 & n & 0 \\ 0 & 0 & 2n \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} (2n)^2 & 0 & 0 \\ 0 & (3n)^2 & 0 \\ 0 & 0 & (4n)^2 \end{pmatrix}$	M1 A1	Or correctly matched permutations of columns.
		8	

Question	Answer	Marks	Guidance
10(i)	$x = -5$ and $y = a$	B1 B1	
		2	
10(ii)	$x^2 + (a+10)x + 5a + 26 = (x+5)(x+a+5) + 1$	M1	By inspection or long division.
	oblique asymptote is $y = x + a + 5$	A1	
		2	
10(iii)	$x^2 + 10x + 5a + 26 = 0$	M1	Puts y–values equal and forms quadratic equation.
	$10^2 - 4(5a+26) = -4 - 20a < 0$ so no intersection point	A1	Correct discriminant and conclusion.
		2	
10(iv)	$(x+5)(2x+a+10) - x^2 - ax - 10x - 5a - 26 = 0$	M1	Differentiates and forms quadratic equation.
	$x^2 + 10x + 24 = 0$	A1	
	Stationary points are $(-4, a+2)$ and $(-6, a-2)$	A1	Must have both points.
		3	

Question	Answer	Marks	Guidance
10(v)		B1	Asymptotes drawn, intersection correct.
		B1	C_1 correct.
		B1	C_2 correct.
		3	

Question	Answer	Marks	Guidance
11E(i)	$x=\sqrt{2}\theta^{\frac{1}{2}}\cos\theta$	M1	Uses $x=r\cos\theta$.
	$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sqrt{2}\theta^{\frac{1}{2}}\cos\theta\right)=\sqrt{2}\left(-\theta^{\frac{1}{2}}\sin\theta+\frac{1}{2}\theta^{-\frac{1}{2}}\cos\theta\right)=0$	M1 A1	Sets derivative of $r\cos\theta$ equal to zero.
	$-\theta^{\frac{1}{2}}\sin\theta+\frac{1}{2}\theta^{-\frac{1}{2}}\cos\theta=0\Rightarrow\cos\theta=2\theta\sin\theta\Rightarrow 2\theta\tan\theta=1$	A1	AG
	$2(0.6)\tan(0.6)-1=-0.179$ and $2(0.7)\tan(0.7)-1=0.179$	B1	Shows sign change.
		5	

Question	Answer	Marks	Guidance
11E(ii)	$2\theta = \theta\sec^2\theta \Rightarrow \theta^{\frac{1}{2}}\left(\sec\theta - \sqrt{2}\right) = 0 \Rightarrow \theta = \frac{\pi}{4}$	M1 A1	Finds value of θ.
		2	
11E(iii)		B1	Correct shape.
		B1	Intersection correct.
		2	

Question	Answer	Marks	Guidance
11E(iv)	$\frac{1}{2}\int_0^{\frac{\pi}{4}} 2\theta\,\mathrm{d}\theta - \frac{1}{2}\int_0^{\frac{\pi}{4}} \theta\sec^2\theta\,\mathrm{d}\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} \theta\left(2-\sec^2\theta\right)\mathrm{d}\theta$	**M1**	Forms correct integral.
	$\frac{1}{2}\left[\theta(2\theta-\tan\theta)\right]_0^{\frac{\pi}{4}} - \frac{1}{2}\int_0^{\frac{\pi}{4}}(2\theta-\tan\theta)\,\mathrm{d}\theta$	**M1 A1**	Integrates by parts.
	$\frac{1}{2}\left[\theta(2\theta-\tan\theta)\right]_0^{\frac{\pi}{4}} - \frac{1}{2}\left[\theta^2+\ln\cos\theta\right]_0^{\frac{\pi}{4}}$	**A1**	
	$\frac{\pi}{8}\left(\frac{\pi}{2}-1\right)-\frac{1}{2}\left(\left(\frac{\pi}{4}\right)^2+\frac{1}{2}\ln 2\right)=\frac{1}{4}\ln 2+\frac{\pi}{8}\left(\frac{\pi}{4}-1\right)$.	**A1**	AEF, must be exact.
		5	
11O(i)(a)	$\begin{pmatrix}-1 & 2 & 3 & 4\\ 1 & 0 & 1 & -1\\ 1 & -2 & -3 & a\\ 1 & 2 & 5 & 2\end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix}-1 & 2 & 3 & 4\\ 0 & 2 & 4 & 3\\ 0 & 0 & 0 & a+4\\ 0 & 0 & 0 & 0\end{pmatrix}$ $v_1\ \ v_2\ \ v_3\ \ v_4 \qquad v_1'\ \ v_2'\ \ v_3'\ \ v_4'$	**M1A1**	Reduces **M** or $\mathbf{M}^{\mathrm{T}}$ to echelon form.
	$\dim V = \text{rank} = 3$.	**A1**	
	$c_1v_1' + c_2v_2' + c_3v_4' = 0 \Rightarrow c_1 = c_2 = c_3 = 0$	**M1**	Shows v_1', v_2', v_4' are linearly independent.
	Thus v_1, v_2, v_4 are linearly independent (and so form a basis for V).	**A1**	
		5	

Question	Answer	Marks	Guidance
11O(i)(b)	$\begin{pmatrix} -1 & 2 & 4 & x \\ 1 & 0 & -1 & y \\ 1 & -2 & a & z \\ 1 & 2 & 2 & t \end{pmatrix} \to \cdots \to \begin{pmatrix} -1 & 2 & 4 & x \\ 0 & 2 & 3 & y+x \\ 0 & 0 & a+4 & z+x \\ 0 & 4 & 6 & t+x \end{pmatrix}$	M1 A1	Uses row operations or $x=-\alpha+2\beta+4\gamma$, $y=\alpha-\gamma$, $z=\alpha-2\beta+a\gamma$, $t=\alpha+2\beta+2\gamma$
	System is consistent when $t+x=2(y+x) \Rightarrow x+2y=t$ Or $(-\alpha+2\beta+4\gamma)+2(\alpha-\gamma)=\alpha+2\beta+2\gamma=t$	M1 A1	AG
		4	
11O(ii)	$-x+2y+3z+4t=0$ $2y+4z+3t=0$	M1	Finds basis for null space.
	$t=\mu,\quad z=\lambda,\quad y=-2\lambda-\frac{3}{2}\mu,\quad x=-\lambda+\mu$	A1	
	A basis is $\left\{\begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 2 \end{pmatrix}\right\} = \{\mathbf{e}_1, \mathbf{e}_2\}$	A1	AEF
	$\mathbf{M}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ so particular solution is $\mathbf{e} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	B1	Finds particular solution.
	General solution is $\mathbf{x} = \mathbf{e} + \lambda\mathbf{e}_1 + \mu\mathbf{e}_2$	A1	**FT** Accept *their* basis. Must have correct particular solution.
		5	

Cambridge International A Level

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CANDIDATE NAME			
CENTRE NUMBER		CANDIDATE NUMBER	

1749707556

FURTHER MATHEMATICS **9231/13**

Paper 1 **May/June 2019**

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF10)

READ THESE INSTRUCTIONS FIRST

Write your centre number, candidate number and name in the spaces at the top of this page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.
DO **NOT** WRITE IN ANY BARCODES.

Answer **all** the questions in the space provided. If additional space is required, you should use the lined page at the end of this booklet. The question number(s) must be clearly shown.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **23** printed pages and **1** blank page.

JC19 06_9231_13/RP

Cambridge Assessment International Education

[Turn over

1 Prove by mathematical induction that $3^{3n} - 1$ is divisible by 13 for every positive integer n. [5]

2 The curve C has polar equation $r^2 = \ln(1 + \theta)$, for $0 \leqslant \theta \leqslant 2\pi$.

(i) Sketch C. [2]

(ii) Using the substitution $u = 1 + \theta$, or otherwise, find the area of the region bounded by C and the initial line, leaving your answer in an exact form. [5]

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3 **(i)** Write down the fifth roots of unity. [2]

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(ii) Find all the roots of the equation

$$z^{10} + z^5 + 1 = 0,$$

giving each root in the form $e^{i\theta}$. [5]

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4 **(i)** Use the method of differences to show that $\sum_{r=1}^{N} \frac{1}{(3r+1)(3r-2)} = \frac{1}{3} - \frac{1}{3(3N+1)}$. [4]

(ii) Find the limit, as $N \to \infty$, of $\displaystyle\sum_{r=N+1}^{N^2} \frac{N}{(3r+1)(3r-2)}$. [4]

5 The linear transformation $\mathrm{T} : \mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix $\mathbf{M}$, where

$$\mathbf{M} = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 5 & 2 & 1 & -3 \\ 4 & 0 & 1 & -7 \\ -2 & 4 & -1 & \alpha \end{pmatrix}.$$

It is given that the rank of $\mathbf{M}$ is 2.

(i) Find the value of α and state a basis for the range space of T. [4]

(ii) Obtain a basis for the null space of T. [4]

6 The curve C has equation

$$y = \frac{x^2}{kx - 1},$$

where k is a positive constant.

(i) Obtain the equations of the asymptotes of C. [3]

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(ii) Find the coordinates of the stationary points of C. [3]

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(iii) Sketch C. [3]

7 The line l_1 passes through the points $A(-3, 1, 4)$ and $B(-1, 5, 9)$. The line l_2 passes through the points $C(-2, 6, 5)$ and $D(-1, 7, 5)$.

(i) Find the shortest distance between the lines l_1 and l_2. [5]

(ii) Find the acute angle between the line l_2 and the plane containing A, B and D. [5]

8 Find the particular solution of the differential equation

$$9\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + x = 50\sin t,$$

given that when $t = 0$, $x = 0$ and $\frac{dx}{dt} = 0$. [10]

9 A cubic equation $x^3 + bx^2 + cx + d = 0$ has real roots α, β and γ such that

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = -\frac{5}{12},$$

$$\alpha\beta\gamma = -12,$$

$$\alpha^3 + \beta^3 + \gamma^3 = 90.$$

(i) Find the values of c and d. [3]

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(ii) Express $\alpha^2 + \beta^2 + \gamma^2$ in terms of b. [2]

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(iii) Show that $b^3 - 15b + 126 = 0$. [4]

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(iv) Given that $3 + i\sqrt{(12)}$ is a root of $y^3 - 15y + 126 = 0$, deduce the value of b. [2]

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10 Let $I_n = \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cot^n x \, dx$, where $n \geqslant 0$.

(i) By considering $\frac{d}{dx}(\cot^{n+1} x)$, or otherwise, show that

$$I_{n+2} = \frac{1}{n+1} - I_n.$$ [5]

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The curve C has equation $y = \cot x$, for $\frac{1}{4}\pi \leqslant x \leqslant \frac{1}{2}\pi$.

(ii) Find, in an exact form, the y-coordinate of the centroid of the region enclosed by C, the line $x = \frac{1}{4}\pi$ and the x-axis. [6]

11 Answer only **one** of the following two alternatives.

EITHER

A 3×3 matrix **A** has distinct eigenvalues 2, 1, 3, with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 0 \\ b \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

respectively, where b is a positive constant.

(i) Find **A** in terms of b. [9]

(ii) Find $\mathbf{A}^{-1}\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$. [2]

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(iii) It is given that

$$\mathbf{A}^n\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{A}^n\begin{pmatrix} -1 \\ 0 \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ b^{-1} \end{pmatrix}.$$

Find the values of n and b. [3]

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OR

The positive variables y and t are related by

$$y = a^t,$$

where a is a positive constant.

(i) (a) By differentiating $\ln y$ with respect to t, show that $\frac{dy}{dt} = a^t \ln a$. [3]

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(b) Write down $\frac{d^2y}{dt^2}$. [1]

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(ii) Determine the set of values of a for which the infinite series

$$y + \frac{dy}{dt} + \frac{d^2y}{dt^2} + \frac{d^3y}{dt^3} + \dots$$

is convergent. [3]

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A curve has parametric equations

$$x = t^a, \quad y = a^t.$$

(iii) Find $\frac{d^2y}{dx^2}$ in terms of a and t, and show that, when $t = 2$,

$$\frac{d^2y}{dx^2} = 2^{1-2a}(1 - a + 2\ln a)\ln a.$$ [7]

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Cambridge International A Level

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FURTHER MATHEMATICS	**9231/13**
Paper 1	**May/June 2019**
MARK SCHEME	
Maximum Mark: 100	

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

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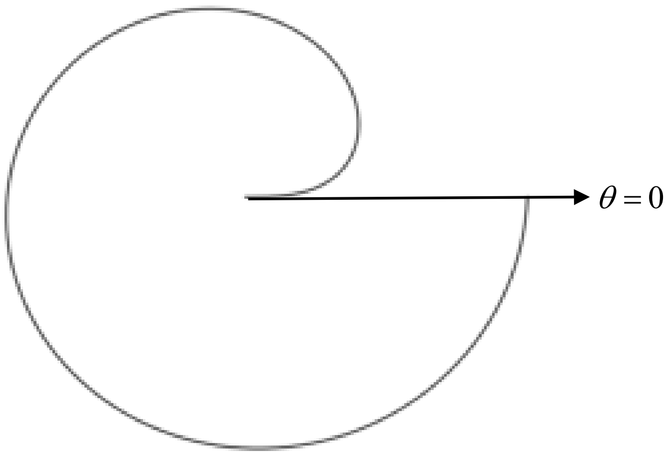
Question	Answer	Marks	Guidance
1	$3^3 - 1 = 26$ is divisible by 13	B1	Checks base case.
	Assume that $3^{3k} - 1$ is divisible by 13 for some positive integer k	B1	States inductive hypothesis.
	Then $3^{3k+3} - 1 = 3^3 3^{3k} - 1 = 26 \cdot 3^{3k} + 3^{3k} - 1$	M1	Separates $3^{3k} - 1$
	is divisible by 13	A1	
	$H_k \Rightarrow H_{k+1}$ By induction, $3^{3n} - 1$ is divisible by 13 for every positive integer n.	A1	States conclusion.
		5	

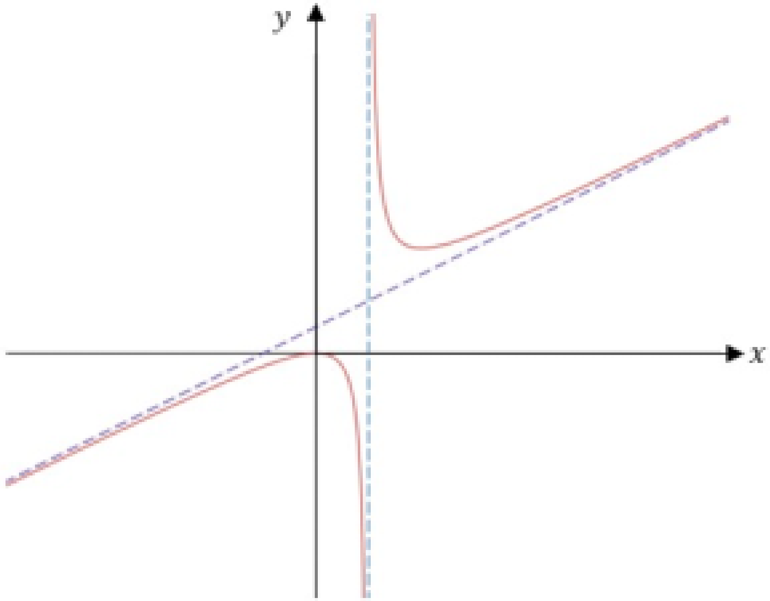
Question	Answer	Marks	Guidance
2(i)		B1	Correct shape, domain and orientation.
		B1	The initial line is tangential to C at the pole.
		2	
2(ii)	$A = \frac{1}{2}\int_0^{2\pi} \ln(1+\theta)\,d\theta$	M1	States $\frac{1}{2}\int r^2\,d\theta$ with correct expression and limits.
	$\int_0^{2\pi} \ln(1+\theta)\,d\theta = \int_1^{2\pi+1} \ln u\ du$	M1	Applies given substitution correctly, changes their limits.
	$\int_1^{2\pi+1} \ln u\ du = [u\ln u]_1^{2\pi+1} - [u]_1^{2\pi+1}$	M1 A1	Integrates $\ln u$ (by parts or otherwise).
	$A = \left(\pi + \frac{1}{2}\right)\ln(2\pi+1) - \pi$	A1	AEF, must be exact.
		5	

Question	Answer	Marks	Guidance
3(i)	$\exp\left(\mathrm{i}\frac{2\pi k}{5}\right)$, $k = 0, \pm 1, \pm 2$	B2	B1 for 1 correct fifth root. B1 for all 5 distinct, correct roots (AEF). SCB1 if only arguments are given.
3(ii)	$z^5 = -\frac{1}{2} \pm \mathrm{i}\frac{\sqrt{3}}{2}$	M1	Correctly solves quadratic in z^5
	$z^5 = \exp\left(\mathrm{i}2\pi\left(\frac{1}{3}+k\right)\right)$ or $\exp\left(\mathrm{i}2\pi\left(-\frac{1}{3}+k\right)\right)$	M1 A1	Writes in polar or exponential form and adds multiples of 2π
	$\exp\left(\pm\mathrm{i}\frac{2\pi}{15}\right)$, $\exp\left(\pm\mathrm{i}\frac{4\pi}{15}\right)$, $\exp\left(\pm\mathrm{i}\frac{8\pi}{15}\right)$, $\exp\left(\pm\mathrm{i}\frac{2\pi}{3}\right)$, $\exp\left(\pm\mathrm{i}\frac{14\pi}{15}\right)$	A2	A1 for 5 distinct, correct roots. A1 for exactly 10 distinct, correct roots. Allow alternative exact values of θ such as $\theta = \frac{16\pi}{15}, \frac{4\pi}{3}, \frac{22\pi}{15}, \frac{26\pi}{15}, \frac{28\pi}{15}$.
	Alternative method for 3(ii)		
	$z^5 + z^{-5} = -1$	M1	Divides through by z^5
	$2\cos 5\theta = -1$	M1 A1	Applies de Moivre's theorem
	$\exp\left(\pm\mathrm{i}\frac{2\pi}{15}\right)$, $\exp\left(\pm\mathrm{i}\frac{4\pi}{15}\right)$, $\exp\left(\pm\mathrm{i}\frac{8\pi}{15}\right)$, $\exp\left(\pm\mathrm{i}\frac{2\pi}{3}\right)$, $\exp\left(\pm\mathrm{i}\frac{14\pi}{15}\right)$	A2	A1 for 5 distinct, correct roots. A1 for exactly 10 distinct, correct roots. Allow alternative exact values of θ such as $\theta = \frac{16\pi}{15}, \frac{4\pi}{3}, \frac{22\pi}{15}, \frac{26\pi}{15}, \frac{28\pi}{15}$.
		5	

Question	Answer	Marks	Guidance
4(i)	$\frac{1}{(3r+1)(3r-2)}=\frac{1}{3}\left(\frac{1}{3r-2}-\frac{1}{3r+1}\right)$	**M1 A1**	Finds partial fractions.
	$\sum_{r=1}^{N}\frac{1}{(3r+1)(3r-2)}=\frac{1}{3}\left(\frac{1}{1}-\frac{1}{4}+\frac{1}{4}-\frac{1}{7}+\ldots\frac{1}{3N-2}-\frac{1}{3N+1}\right)$	**M1**	At least 3 term including final term.
	$=\frac{1}{3}\left(1-\frac{1}{3N+1}\right)$	**A1**	AG
		4	
4(ii)	$\sum_{r=N+1}^{N^2}\frac{N}{(3r+1)(3r-2)}=\sum_{r=1}^{N^2}\frac{N}{(3r+1)(3r-2)}-\sum_{r=1}^{N}\frac{N}{(3r+1)(3r-2)}$	**M1**	Uses $\sum_{r=N+1}^{N^2}=\sum_{r=1}^{N^2}-\sum_{r=1}^{N}$
	$=\frac{N}{3}-\frac{N}{3(3N^2+1)}-\left(\frac{N}{3}-\frac{N}{3(3N+1)}\right)$	**M1**	Applies **(i)**
	$=\frac{N}{3(3N+1)}-\frac{N}{3(3N^2+1)}=\frac{N^3-N^2}{(3N+1)(3N^2+1)}$	**A1**	Allow simplification to common denominator.
	$\rightarrow\frac{1}{9}$ as $N\rightarrow\infty$	**B1**	
		4	

Question	Answer	Marks	Guidance
5(i)	$\begin{pmatrix}1&2&0&4\\5&2&1&-3\\4&0&1&-7\\-2&4&-1&\alpha\end{pmatrix}\to\begin{pmatrix}1&2&0&4\\0&-8&1&-23\\0&-8&1&-23\\0&8&-1&\alpha+8\end{pmatrix}\to\begin{pmatrix}1&2&0&4\\0&-8&1&-23\\0&0&0&0\\0&0&0&\alpha-15\end{pmatrix}$	M1 A1	Performs row operations correctly.
	$r(\mathbf{M})=2\Rightarrow\alpha=15$	A1	CWO
	Basis for the range space is $\left\{\begin{pmatrix}1\\5\\4\\-2\end{pmatrix},\begin{pmatrix}2\\2\\0\\4\end{pmatrix}\right\}$	B1	Accept any two independent column vectors from **M**.
		4	
5(ii)	$x+2y\quad+4t=0$ $-8y+z-23t=0$	M1	Forms system of equations.
	$t=\lambda,\ z=\mu,\ y=\frac{1}{8}\mu-\frac{23}{8}\lambda,\ x=-\frac{1}{4}\mu+\frac{7}{4}\lambda$	M1	Uses two parameters.
	Basis for null space is $\left\{\begin{pmatrix}-2\\1\\8\\0\end{pmatrix},\begin{pmatrix}14\\-23\\0\\8\end{pmatrix}\right\}$	A1 A1	AEF. Many alternatives are possible e.g. $\left\{\begin{pmatrix}-2\\1\\8\\0\end{pmatrix},\begin{pmatrix}-4\\0\\23\\1\end{pmatrix}\right\},\left\{\begin{pmatrix}4\\0\\-23\\-1\end{pmatrix},\begin{pmatrix}0\\2\\-7\\-1\end{pmatrix}\right\}$
		4	

Question	Answer	Marks	Guidance
6(i)	$x=\frac{1}{k}$	**B1**	States vertical asymptote.
	$y=\frac{(kx-1)\left(k^{-1}x+k^{-2}\right)+k^{-2}}{kx-1}$	**M1**	Finds oblique asymptote.
	Oblique asymptote is $y=k^{-1}x+k^{-2}$	**A1**	
		3	
6(ii)	$y'=\frac{2x(kx-1)-kx^2}{(kx-1)^2}=0 \Rightarrow kx^2-2x=0$	**M1**	Differentiates and equates to 0.
	$x=0,\ 2k^{-1}$	**A1**	Finds x-coordinates.
	$(0,0),\ \left(2k^{-1},4k^{-2}\right)$	**A1**	Finds y-coordinates
		3	

Question	Answer	Marks	Guidance
6(iii)		B1	Axes and asymptotes correct.
		B1	Upper branch correct.
		B1	Lower branch correct. Deduct at most 1 mark for poor forms at infinity.
		3	

Question	Answer	Marks	Guidance
7(i)	$\overrightarrow{AB}=\begin{pmatrix}2\\4\\5\end{pmatrix},\ \overrightarrow{CD}=\begin{pmatrix}1\\1\\0\end{pmatrix}$	**B1**	Find the directions of the lines.
	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&4&5\\1&1&0\end{vmatrix}=\begin{pmatrix}-5\\5\\-2\end{pmatrix}$	**M1 A1**	Finds direction of common perpendicular. Allow any non-zero scalar multiple.
	$\dfrac{\begin{pmatrix}1\\5\\1\end{pmatrix}\cdot\begin{pmatrix}-5\\5\\-2\end{pmatrix}}{\sqrt{5^2+5^2+2^2}}=\dfrac{18}{\sqrt{54}}=\sqrt{6}=2.45$	**M1 A1**	Uses formula for shortest distance.
		5	
7(ii)	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&4&5\\2&6&1\end{vmatrix}=\begin{pmatrix}-26\\8\\4\end{pmatrix}\sim\begin{pmatrix}-13\\4\\2\end{pmatrix}$	**M1 A1**	Finds normal to the plane.
	$\dfrac{\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}-13\\4\\2\end{pmatrix}}{\sqrt{1^2+1^2+0^2}\sqrt{13^2+4^2+2^2}}=-\dfrac{9}{\sqrt{378}}=-\dfrac{\sqrt{42}}{14}$	**M1 A1**	**FT** Uses dot product of their normal with direction of line.
	$\cos^{-1}\left(-\dfrac{\sqrt{42}}{14}\right)-90=27.6°\quad(\text{or } 0.481\,\text{rad})$	**A1**	
		5	

Question	Answer	Marks	Guidance
8	$9u^2+6u+1=0 \Rightarrow (3u+1)^2=0$	M1	Auxiliary equation.
	CF: $x=(At+B)e^{-\frac{1}{3}t}$	A1	States CF.
	$x=p\sin t+q\cos t \Rightarrow \dot{x}=p\cos t-q\sin t$ $\Rightarrow \ddot{x}=-p\sin t-q\cos t$	M1	Forms PI and differentiates.
	$-9p-6q+p=50$ $-9q+6p+q=0$	M1	Substitutes.
	$q=-3, p=-4.$	A1	
	$x=(At+B)e^{-\frac{1}{3}t}-4\sin t-3\cos t$	A1	States general solution. FT if correct form of CF/PI.
	$\dot{x}=-\frac{1}{3}(At+B)e^{-\frac{1}{3}t}+Ae^{-\frac{1}{3}t}-4\cos t+3\sin t$	M1	Differentiates and forms equations using initial conditions.
	$B=3$	B1	
	$-\frac{1}{3}B+A-4=0 \Rightarrow A=5$	A1	.
	$x=(5t+3)e^{-\frac{1}{3}t}-4\sin t-3\cos t$	A1	States PS.
		10	

Question	Answer	Marks	Guidance
9(i)	$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\alpha\beta+\beta\gamma+\alpha\gamma}{\alpha\beta\gamma}\Rightarrow c=\alpha\beta+\beta\gamma+\alpha\gamma=5$	M1 A1	Uses $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\alpha\beta+\beta\gamma+\alpha\gamma}{\alpha\beta\gamma}$.
	$d=12$	B1	
		3	
9(ii)	$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$	M1	
	$(\alpha+\beta+\gamma)^2-10=b^2-10$	A1	
	Alternative method for 9(ii)		
	$90+bS_2-5b+36=0\Rightarrow S_2=\frac{5b-126}{b}$	M1 A1	Sums over roots
		2	

Question	Answer	Marks	Guidance
9(iii)	$x^3 + bx^2 + 5x + 12 = 0$ $\quad (-b = \alpha + \beta + \gamma)$	M1	Formulates equation.
	$90 + b\left(b^2 - 10\right) - 5b + 36 = 0$	M1 A1	Sums over roots and uses **9(ii)**
	$b^3 - 15b + 126 = 0$	A1	
	Alternative method 1 for 9(iii)		
	$(\alpha+\beta+\gamma)^3 = \alpha^3 + \beta^3 + \gamma^3 + 3(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha) - 3\alpha\beta\gamma$	M1 A1	Uses expansion of $(\alpha+\beta+\gamma)^3$
	$-b^3 = 90 - 15b + 36$	(M1)	Substitutes known values.
	$b^3 - 15b + 126 = 0$	A1	
	Alternative method 2 for 9(iii)		
	$\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha) = b^2 - 10$	M1 A1	Finds sum of squares in terms of *b*.
	$\dfrac{5b-126}{b} = b^2 - 10 \Rightarrow b^3 - 15b + 126 = 0$	M1 A1	Equates expressions.
		4	
9(iv)	Real root is $b = -6$	M1 A1	Gives the real root for b.

Question	Answer	Marks	Guidance
10(i)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cot^{n+1}x\right) = -(n+1)\cot^{n}x\csc^{2}x$	M1 A1	Differentiates using chain rule.
	$= -(n+1)\cot^{n}x\left(\cot^{2}x+1\right)$ $= -(n+1)\cot^{n+2}x-(n+1)\cot^{n}x$	M1	Uses $\csc^{2}x = \cot^{2}x+1$. OE
	$\left[\cot^{n+1}x\right]_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} = -(n+1)\left(I_{n+2}+I_{n}\right)$	M1	Integrates both sides.
	$-1 = -(n+1)\left(I_{n+2}+I_{n}\right) \Rightarrow I_{n+2} = \frac{1}{n+1}-I_{n}$	A1	AG
10(i)	**Alternative method for 10(i)**		
	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\cos^{n+1}x}{\sin^{n+1}x}\right) = \frac{-(n+1)\sin^{n+2}x\cos^{n}x-(n+1)\sin^{n}x\cos^{n+2}x}{\sin^{2n+2}x}$	M1 A1	Differentiates using quotient/chain rule.
	$-(n+1)\cot^{n}x-(n+1)\cot^{n+2}x$	M1	Separates
	$\left[\cot^{n+1}x\right]_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} = -(n+1)\left(I_{n+2}+I_{n}\right)$	M1	Integrates both sides.
	$-1 = -(n+1)\left(I_{n+2}+I_{n}\right) \Rightarrow I_{n+2} = \frac{1}{n+1}-I_{n}$	A1	AG
		5	

Question	Answer	Marks	Guidance
10(ii)	$\bar{y} = \frac{1}{2A}\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cot^2 x \, dx = \frac{I_2}{2A}$	M1	Uses correct formula for $\bar{y}$. Allow incorrect or missing limits.
	$A = \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cot x \, dx = [\ln \sin x]_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} = \ln\sqrt{2}$	M1 A1	Integrates $\cot x$ to find area under the curve with correct limits.
	$I_2 = 1 - I_0 = 1 - \frac{1}{4}\pi$	M1 A1	Finds I_2
	$\bar{y} = \frac{1}{\ln 2}\left(1 - \frac{1}{4}\pi\right)$	A1	AEF. Must be exact.
		6	

Question	Answer	Marks	Guidance
11E(i)	$\mathbf{P}=\begin{pmatrix}1&-1&0\\1&0&1\\0&b&-1\end{pmatrix}$ $\mathbf{D}=\begin{pmatrix}2&0&0\\0&1&0\\0&0&3\end{pmatrix}$	B1 B1	Writes **P** and **D** (accept correctly matched permutations of columns)
	$\det\mathbf{P}=-b-1$	B1	
	$\mathbf{P}^{-1}=\frac{1}{b+1}\begin{pmatrix}b&-1&-b\\1&1&b\\1&1&-1\end{pmatrix}^{\mathrm{T}}=\frac{1}{b+1}\begin{pmatrix}b&1&1\\-1&1&1\\-b&b&-1\end{pmatrix}$	M1 A1	Finds inverse of **P**. (Adj ÷ Det).
	$\mathbf{A}=\mathbf{PDP}^{-1}$	M1	Applies $\mathbf{A}=\mathbf{PDP}^{-1}$
	$\mathbf{A}=\frac{1}{b+1}\begin{pmatrix}1&-1&0\\1&0&1\\0&b&-1\end{pmatrix}\begin{pmatrix}2b&2&2\\-1&1&1\\-3b&3b&-3\end{pmatrix}$	M1 A1	Multiplies two adjacent matrices. $\mathbf{PD}=\begin{pmatrix}2&-1&0\\2&0&3\\0&b&-3\end{pmatrix}$
	$\frac{1}{b+1}\begin{pmatrix}1+2b&1&1\\-b&3b+2&-1\\2b&-2b&3+b\end{pmatrix}$	A1	

Question	Answer	Marks	Guidance
11E(i)	**Alternative method for 11E(i)**		
	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{22} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$ $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{22} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ b \end{pmatrix}$ $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{22} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix}$	**B3**	Matches eigenvalues with eigenvectors to form systems of equations. B1 for each correctly matched pair.
	Three values found correctly from solving systems of equations. Another three values found correctly from solving systems of equations. Attempting to find final three values from systems of equations.	**M1 A1** **M1 A1** **M1**	
	$\mathbf{A} = \frac{1}{b+1}\begin{pmatrix} 1+2b & 1 & 1 \\ -b & 3b+2 & -1 \\ 2b & -2b & 3+b \end{pmatrix}$	**A1**	Final answer given in matrix form.
		9	
11E(ii)	$\mathbf{A}^{-1}\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} = 2\mathbf{A}^{-1}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \frac{2}{3}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$	**M1 A1**	M1 for $\mathbf{A}^{-1}\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} = 2\mathbf{A}^{-1}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.
		2	

Question	Answer	Marks	Guidance
11E(iii)	$\mathbf{A}^n\begin{pmatrix}1\\1\\0\end{pmatrix}=2^2\begin{pmatrix}1\\1\\0\end{pmatrix}\Rightarrow n=2$	B1	Uses $\mathbf{A}^n\begin{pmatrix}1\\1\\0\end{pmatrix}=2^n\begin{pmatrix}1\\1\\0\end{pmatrix}$
	$\mathbf{A}^n\begin{pmatrix}-1\\0\\b\end{pmatrix}=\begin{pmatrix}-1\\0\\b^{-1}\end{pmatrix}\Rightarrow b=b^{-1}\Rightarrow b=1$	M1 A1	M1 for stating $b=b^{-1}$ (since $\mathbf{A}^n\begin{pmatrix}-1\\0\\b\end{pmatrix}=\begin{pmatrix}-1\\0\\b\end{pmatrix}$). Must be only one answer ($b=1$) for A1.
		3	

Question	Answer	Marks	Guidance
11O(i)(a)	$\ln y = t \ln a$ $\frac{1}{y}\frac{dy}{dt} = \ln a$	M1 A1	Differentiates both sides.
	$\frac{dy}{dt} = y \ln a = a^t \ln a$	A1	AG
		3	
11O(i)(b)	$\frac{d^2 y}{dt^2} = a^t (\ln a)^2$	B1	
11O(ii)	$r = \ln a$	B1	States common ratio.
	$-1 < \ln a < 1 \Rightarrow e^{-1} < a < e$	M1 A1	Uses convergence condition for a geometric series.
		3	

Question	Answer	Marks	Guidance
11O(iii)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\times\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{a^t\ln a}{at^{a-1}}=\left(\frac{\ln a}{a}\right)\left(\frac{a^t}{t^{a-1}}\right)$	M1 A1	Uses chain rule to find $\frac{\mathrm{d}y}{\mathrm{d}x}$
	$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\left(\frac{\ln a}{a}\right)\left(\frac{t^{a-1}a^t\ln a-(a-1)t^{a-2}a^t}{t^{2(a-1)}}\right)$	M1 A1	Uses quotient or product rule to find $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$
	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\left(\frac{\ln a}{a^2}\right)\left(\frac{t^{a-1}a^t\ln a-(a-1)t^{a-2}a^t}{t^{3(a-1)}}\right)$ $=\left(\frac{\ln a}{a^2}\right)\left(\frac{\ln a-(a-1)t^{-1}}{t^{2(a-1)}}\right)a^t$	M1 A1	Uses chain rule to find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$
	$t=2\Rightarrow\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2^{1-2a}\ln a(2\ln a-a+1)$	A1	Substitutes $t=2$, AG.
		7	