Name: ____________________ **Date:** ____________ **Class:** __________

Past Paper Questions: Mathematical Induction

1 Prove by induction that, for all $N \geqslant 1$,

$$\sum_{n=1}^{N} \frac{n+2}{n(n+1)2^n} = 1 - \frac{1}{(N+1)2^N}.$$ [5]

2 It is given that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{\ln x}{x}\right) = \frac{a_n \ln x + b_n}{x^{n+1}},$$

where a_n and b_n depend only on n.

(i) Find a_1, a_2 and a_3. [3]

(ii) Use mathematical induction to establish a formula for a_n. [5]

3 The integral I_n, where n is a non-negative integer, is defined by

$$I_n = \int_0^1 e^{-x}(1-x)^n \, dx.$$

(i) Show that $I_{n+1} = 1 - (n+1)I_n$. [3]

(ii) Use induction to show that I_n is of the form $A_n + B_n e^{-1}$, where A_n and B_n are integers. [4]

(iii) Express B_n in terms of n. [2]

4 Prove by induction, or otherwise, that

$$23^{2n} + 31^{2n} + 46$$

is divisible by 48, for all integers $n \geqslant 0$. [6]

5 The sequence $x_1, x_2, x_3, \ldots$ is such that $x_1 = 1$ and

$$x_{n+1} = \frac{1 + 4x_n}{5 + 2x_n}.$$

Prove by induction that $x_n > \frac{1}{2}$ for all $n \geqslant 1$. [5]

Prove also that $x_n > x_{n+1}$ for all $n \geqslant 1$. [3]

6 Prove by induction that

$$\sum_{r=1}^{n}(3r^5 + r^3) = \tfrac{1}{2}n^3(n+1)^3,$$

for all $n \geqslant 1$. [5]

Use this result together with the List of Formulae (MF10) to prove that

$$\sum_{r=1}^{n} r^5 = \tfrac{1}{12}n^2(n+1)^2\mathrm{Q}(n),$$

where $\mathrm{Q}(n)$ is a quadratic function of n which is to be determined. [3]

7 Let

$$I_n = \int_0^1 t^n e^{-t}\,dt,$$

where $n \geqslant 0$. Show that, for all $n \geqslant 1$,

$$I_n = nI_{n-1} - e^{-1}.$$ [3]

Hence prove by induction that, for all positive integers n,

$$I_n < n!.$$ [5]

8 Let

$$I_n = \int_1^{e} x(\ln x)^n \, dx,$$

where $n \geqslant 1$. Show that

$$I_{n+1} = \tfrac{1}{2}e^2 - \tfrac{1}{2}(n+1)I_n. \qquad [3]$$

Hence prove by induction that, for all positive integers n, I_n is of the form $A_n e^2 + B_n$, where A_n and B_n are rational numbers. [6]

9 The sequence $x_1, x_2, x_3, \ldots$ is such that $x_1 = 3$ and

$$x_{n+1} = \frac{2x_n^2 + 4x_n - 2}{2x_n + 3}$$

for $n = 1, 2, 3, \ldots$. Prove by induction that $x_n > 2$ for all n. [6]

10 It is given that $f(n) = 3^{3n} + 6^{n-1}$.

(i) Show that $f(n+1) + f(n) = 28(3^{3n}) + 7(6^{n-1})$. [2]

(ii) Hence, or otherwise, prove by mathematical induction that $f(n)$ is divisible by 7 for every positive integer n. [4]

11 Let $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$. Prove by mathematical induction that, for every positive integer n,

$$\mathbf{A}^n = \begin{pmatrix} 2^n & 3(2^n - 1) \\ 0 & 1 \end{pmatrix}.$$

[5]

12 Prove, by mathematical induction, that, for integers $n \geqslant 2$,

$$4^n > 2^n + 3^n.$$ [5]

13 For the sequence $u_1, u_2, u_3, \ldots$, it is given that $u_1 = 1$ and $u_{r+1} = \frac{3u_r - 2}{4}$ for all r. Prove by mathematical induction that $u_n = 4\left(\frac{3}{4}\right)^n - 2$, for all positive integers n. [5]

14 Prove by mathematical induction that $5^{2n} - 1$ is divisible by 8 for every positive integer n. [5]

15 Prove by mathematical induction that, for every positive integer n,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^x \sin x) = (\sqrt{2})^n \mathrm{e}^x \sin(x + \tfrac{1}{4}n\pi).$$ [7]

16 Prove by mathematical induction that, for all non-negative integers n,

$$11^{2n} + 25^n + 22$$

is divisible by 24. [6]

17 It is given that $\phi(n) = 5^n(4n + 1) - 1$, for $n = 1, 2, 3, \ldots$. Prove, by mathematical induction, that $\phi(n)$ is divisible by 8, for every positive integer n. [7]

18 The sequence $a_1, a_2, a_3, \ldots$ is such that $a_1 > 5$ and $a_{n+1} = \frac{4a_n}{5} + \frac{5}{a_n}$ for every positive integer n. Prove by mathematical induction that $a_n > 5$ for every positive integer n. [5]

Prove also that $a_n > a_{n+1}$ for every positive integer n. [2]

19 Prove by mathematical induction that, for all positive integers n, $\sum_{r=1}^{n} \frac{1}{(2r)^2 - 1} = \frac{n}{2n+1}$. [6]

State the value of $\sum_{r=1}^{\infty} \frac{1}{(2r)^2 - 1}$. [1]

20 Prove by mathematical induction that, for all positive integers n, $10^n + 3 \times 4^{n+2} + 5$ is divisible by 9. [6]

21 It is given that a diagonal of a polygon is a line joining two non-adjacent vertices. Prove, by mathematical induction, that an n-sided polygon has $\frac{1}{2}n(n-3)$ diagonals, where $n \geqslant 3$. [6]

22 Prove, by mathematical induction, that $5^n + 3$ is divisible by 4 for all non-negative integers n. [5]

23 Prove, by mathematical induction, that $\sum_{r=1}^{n} r \ln\left(\frac{r+1}{r}\right) = \ln\left(\frac{(n+1)^n}{n!}\right)$ for all positive integers n. [6]

24 It is given that $f(n) = 2^{3n} + 8^{n-1}$. By simplifying $f(k) + f(k+1)$, or otherwise, prove by mathematical induction that $f(n)$ is divisible by 9 for every positive integer n. [6]

25 For the sequence u_1, u_2, u_3, …, it is given that $u_1 = 8$ and

$$u_{r+1} = \frac{5u_r - 3}{4}$$

for all r.

(i) Prove by mathematical induction that

$$u_n = 4\left(\tfrac{5}{4}\right)^n + 3,$$

for all positive integers n. [5]

(ii) Deduce the set of values of x for which the infinite series

$$(u_1 - 3)x + (u_2 - 3)x^2 + \dots + (u_r - 3)x^r + \dots$$

is convergent. [2]

(iii) Use the result given in part **(i)** to find surds a and b such that

$$\sum_{n=1}^{N} \ln(u_n - 3) = N^2 \ln a + N \ln b.$$ [3]

26 **(i)** Prove by mathematical induction that, for $z \neq 1$ and all positive integers n,

$$1 + z + z^2 + \dots + z^{n-1} = \frac{z^n - 1}{z - 1}.$$ [5]

(ii) By letting $z = \frac{1}{2}(\cos\theta + \mathrm{i}\sin\theta)$, use de Moivre's theorem to deduce that

$$\sum_{m=1}^{\infty} \left(\tfrac{1}{2}\right)^m \sin m\theta = \frac{2\sin\theta}{5 - 4\cos\theta}.$$ [5]

27 Prove by mathematical induction that $3^{3n} - 1$ is divisible by 13 for every positive integer n. [5]

28 The sequence of real numbers $a_1, a_2, a_3, \ldots$ is such that $a_1 = 1$ and

$$a_{n+1} = \left(a_n + \frac{1}{a_n}\right)^{\lambda},$$

where λ is a constant greater than 1. Prove by mathematical induction that, for $n \geqslant 2$,

$$a_n \geqslant 2^{g(n)},$$

where $g(n) = \lambda^{n-1}$. [6]

Prove also that, for $n \geqslant 2$, $\dfrac{a_{n+1}}{a_n} > 2^{(\lambda-1)g(n)}$. [3]

29 The sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = 1$ and

$$u_{n+1} = -1 + \sqrt{(u_n + 7)}.$$

(i) Prove by induction that $u_n < 2$ for all $n \geqslant 1$. [4]

(ii) Show that if $u_n = 2 - \varepsilon$, where ε is small, then

$$u_{n+1} \approx 2 - \tfrac{1}{6}\varepsilon.$$ [2]

30 Prove by mathematical induction that, for all positive integers n, $10^{3n} + 13^{n+1}$ is divisible by 7. [5]

31 Prove by induction that, for all $n \geqslant 1$,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\mathrm{e}^{x^2}\right) = \mathrm{P}_n(x)\mathrm{e}^{x^2},$$

where $\mathrm{P}_n(x)$ is a polynomial in x of degree n with the coefficient of x^n equal to 2^n. [6]

32 Prove by mathematical induction that, for all non-negative integers n, $7^{2n+1} + 5^{n+3}$ is divisible by 44. [5]

33 Prove by mathematical induction that, for all positive integers n,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^x \sin x) = 2^{\frac{1}{2}n}\mathrm{e}^x \sin\left(x + \tfrac{1}{4}n\pi\right).$$ [7]

34 Prove by mathematical induction that, for all positive integers n,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{1}{2x+3}\right) = (-1)^n \frac{n!\,2^n}{(2x+3)^{n+1}}.$$ [6]

35 Let I_n denote $\int_0^{\infty} x^n e^{-2x}\, dx$. Show that $I_n = \frac{1}{2} n I_{n-1}$, for $n \geqslant 1$. [2]

Prove by mathematical induction that, for all positive integers n, $I_n = \dfrac{n!}{2^{n+1}}$. [6]

36 Let $S_N = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{N}{(N+1)!}$. Prove by mathematical induction that, for all positive integers N,

$$S_N = 1 - \frac{1}{(N+1)!}.$$

[5]

37 It is given that $y = (1 + x)^2 \ln(1 + x)$. Find $\dfrac{d^3y}{dx^3}$. [3]

Prove by mathematical induction that, for every integer $n \geqslant 3$,

$$\frac{d^ny}{dx^n} = (-1)^{n-1}\frac{2(n-3)!}{(1+x)^{n-2}}.$$ [5]

38 Prove by mathematical induction that, for every positive integer n,

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta. \qquad [5]$$

Express $\sin^5\theta$ in the form $p\sin 5\theta + q\sin 3\theta + r\sin\theta$, where p, q and r are rational numbers to be determined. [6]

39 It is given that $u_r = r \times r!$ for $r = 1, 2, 3, \ldots$. Let $S_n = u_1 + u_2 + u_3 + \ldots + u_n$. Write down the values of

$$2! - S_1, \quad 3! - S_2, \quad 4! - S_3, \quad 5! - S_4.$$ [2]

Conjecture a formula for S_n. [1]

Prove, by mathematical induction, a formula for S_n, for all positive integers n. [4]

40 Given that a is a constant, prove by mathematical induction that, for every positive integer n,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x\mathrm{e}^{ax}) = na^{n-1}\mathrm{e}^{ax} + a^n x\mathrm{e}^{ax}.$$ [6]

41 Using factorials, show that $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$. [2]

Hence prove by mathematical induction that

$$(a + x)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}x + \ldots + \binom{n}{r}a^{n-r}x^r + \ldots + \binom{n}{n}x^n$$

for every positive integer n. [4]

42 **(i)** Show that $\dfrac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}(x^{n+1}\ln x) = \dfrac{\mathrm{d}^n}{\mathrm{d}x^n}\big(x^n + (n+1)x^n \ln x\big)$. [2]

(ii) Prove by mathematical induction that, for all positive integers n,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n \ln x) = n!\left(\ln x + 1 + \frac{1}{2} + \dots + \frac{1}{n}\right).$$ [5]

43 The sequence of positive numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 < 3$ and, for $n \geqslant 1$,

$$u_{n+1} = \frac{4u_n + 9}{u_n + 4}.$$

(i) By considering $3 - u_{n+1}$, or otherwise, prove by mathematical induction that $u_n < 3$ for all positive integers n. [5]

(ii) Show that $u_{n+1} > u_n$ for $n \geqslant 1$. [3]

44 It is given that $y = \mathrm{e}^x u$, where u is a function of x. The rth derivatives $\dfrac{\mathrm{d}^r y}{\mathrm{d}x^r}$ and $\dfrac{\mathrm{d}^r u}{\mathrm{d}x^r}$ are denoted by $y^{(r)}$ and $u^{(r)}$ respectively. Prove by mathematical induction that, for all positive integers n,

$$y^{(n)} = \mathrm{e}^x\left(\binom{n}{0}u + \binom{n}{1}u^{(1)} + \binom{n}{2}u^{(2)} + \ldots + \binom{n}{r}u^{(r)} + \ldots + \binom{n}{n}u^{(n)}\right).$$ [8]

[You may use without proof the result $\binom{k}{r} + \binom{k}{r-1} = \binom{k+1}{r}$.]

45 It is given that $y = \ln(ax + 1)$, where a is a positive constant. Prove by mathematical induction that, for every positive integer n,

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = (-1)^{n-1} \frac{(n-1)!a^n}{(ax+1)^n}.$$

[6]

46 Prove by mathematical induction that, for every positive integer n,

$$\frac{\mathrm{d}^{2n-1}}{\mathrm{d}x^{2n-1}}(x\sin x) = (-1)^{n-1}\big(x\cos x + (2n-1)\sin x\big).$$ [7]

47 Prove by mathematical induction that $7^{2n} - 1$ is divisible by 12 for every positive integer n. [5]

48 The sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = 1$ and $u_{n+1} = 2u_n + 1$ for $n \geqslant 1$.

(a) Prove by induction that $u_n = 2^n - 1$ for all positive integers n. [5]

(b) Deduce that u_{2n} is divisible by u_n for $n \geqslant 1$. [2]

49 Prove by mathematical induction that $2^{4n} + 31^n - 2$ is divisible by 15 for all positive integers n. [6]

50 **(a)** Prove by mathematical induction that, for all positive integers n,

$$\sum_{r=1}^{n}\left(5r^4 + r^2\right) = \tfrac{1}{2}n^2(n+1)^2(2n+1).$$

(b) Use the result given in part **(a)** together with the List of formulae (MF19) to find $\sum_{r=1}^{n} r^4$ in terms of n, fully factorising your answer. [3]

51 The sequence of positive numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 > 4$ and, for $n \geqslant 1$,

$$u_{n+1} = \frac{u_n^2 + u_n + 12}{2u_n}.$$

(a) By considering $u_{n+1} - 4$, or otherwise, prove by mathematical induction that $u_n > 4$ for all positive integers n. [5]

(b) Show that $u_{n+1} < u_n$ for $n \geqslant 1$. [3]

52 Let $\mathbf{A} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, where a is a positive constant.

(a) State the type of the geometrical transformation in the x–y plane represented by $\mathbf{A}$. [1]

(b) Prove by mathematical induction that, for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}.$$ [5]

Let $\mathbf{B} = \begin{pmatrix} b & b \\ a^{-1} & a^{-1} \end{pmatrix}$, where b is a positive constant.

(c) Find the equations of the invariant lines, through the origin, of the transformation in the x–y plane represented by $\mathbf{A}^n\mathbf{B}$. [6]

53 The sequence of real numbers $a_1, a_2, a_3, \ldots$ is such that $a_1 = 1$ and

$$a_{n+1} = \left(a_n + \frac{1}{a_n}\right)^3.$$

(a) Prove by mathematical induction that $\ln a_n \geqslant 3^{n-1} \ln 2$ for all integers $n \geqslant 2$. [6]

[You may use the fact that $\ln\left(x + \frac{1}{x}\right) > \ln x$ for $x > 0$.]

(b) Show that $\ln a_{n+1} - \ln a_n > 3^{n-1} \ln 4$ for $n \geqslant 2$. [2]

54 It is given that $y = xe^{ax}$, where a is a constant.

Prove by mathematical induction that, for all positive integers n,

$$\frac{d^n y}{dx^n} = \left(a^n x + na^{n-1}\right)e^{ax}.$$ [6]

55 Prove by mathematical induction that, for all positive integers n, $7^{2n} + 97^n - 50$ is divisible by 48. [6]

56 The function f is such that $f''(x) = f(x)$.

Prove by mathematical induction that, for every positive integer n,

$$\frac{d^{2n-1}}{dx^{2n-1}}\big(x f(x)\big) = x f'(x) + (2n-1) f(x).$$ [7]