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## Past Paper Questions: Mathematical Induction

1 Prove by induction that, for all  $N \ge 1$ ,

$$\sum_{n=1}^{N} \frac{n+2}{n(n+1)2^n} = 1 - \frac{1}{(N+1)2^N}.$$
 [5]

## 2 It is given that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \left( \frac{\ln x}{x} \right) = \frac{a_n \ln x + b_n}{x^{n+1}},$$

where  $a_n$  and  $b_n$  depend only on n.

(i) Find 
$$a_1, a_2$$
 and  $a_3$ . [3]

(ii) Use mathematical induction to establish a formula for  $a_n$ . [5]

3 The integral  $I_n$ , where n is a non-negative integer, is defined by

$$I_n = \int_0^1 e^{-x} (1-x)^n dx.$$

- (i) Show that  $I_{n+1} = 1 (n+1)I_n$ . [3]
- (ii) Use induction to show that  $I_n$  is of the form  $A_n + B_n e^{-1}$ , where  $A_n$  and  $B_n$  are integers. [4]
- (iii) Express  $B_n$  in terms of n. [2]

4 Prove by induction, or otherwise, that

$$23^{2n} + 31^{2n} + 46$$

is divisible by 48, for all integers  $n \ge 0$ .

[6]

5 The sequence  $x_1, x_2, x_3, \dots$  is such that  $x_1 = 1$  and

$$x_{n+1} = \frac{1 + 4x_n}{5 + 2x_n}.$$

Prove by induction that 
$$x_n > \frac{1}{2}$$
 for all  $n \ge 1$ . [5]

Prove also that 
$$x_n > x_{n+1}$$
 for all  $n \ge 1$ . [3]

## 6 Prove by induction that

$$\sum_{r=1}^{n} (3r^5 + r^3) = \frac{1}{2}n^3(n+1)^3,$$

for all  $n \ge 1$ . [5]

Use this result together with the List of Formulae (MF10) to prove that

$$\sum_{r=1}^{n} r^5 = \frac{1}{12} n^2 (n+1)^2 Q(n),$$

[3]

where Q(n) is a quadratic function of n which is to be determined.

7 Let

$$I_n = \int_0^1 t^n \mathrm{e}^{-t} \, \mathrm{d}t,$$

where  $n \ge 0$ . Show that, for all  $n \ge 1$ ,

$$I_n = nI_{n-1} - e^{-1}$$
. [3]

Hence prove by induction that, for all positive integers n,

$$I_n < n!. ag{5}$$

## 8 Let

$$I_n = \int_1^e x(\ln x)^n \, \mathrm{d}x,$$

where  $n \ge 1$ . Show that

$$I_{n+1} = \frac{1}{2}e^2 - \frac{1}{2}(n+1)I_n.$$
 [3]

Hence prove by induction that, for all positive integers n,  $I_n$  is of the form  $A_n e^2 + B_n$ , where  $A_n$  and  $B_n$  are rational numbers. [6]

9 The sequence  $x_1, x_2, x_3, \dots$  is such that  $x_1 = 3$  and

$$x_{n+1} = \frac{2x_n^2 + 4x_n - 2}{2x_n + 3}$$

[6]

for 
$$n = 1, 2, 3, \ldots$$
. Prove by induction that  $x_n > 2$  for all  $n$ .

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- 10 It is given that  $f(n) = 3^{3n} + 6^{n-1}$ .
  - (i) Show that  $f(n+1) + f(n) = 28(3^{3n}) + 7(6^{n-1})$ . [2]
  - (ii) Hence, or otherwise, prove by mathematical induction that f(n) is divisible by 7 for every positive integer n. [4]

11 Let  $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$ . Prove by mathematical induction that, for every positive integer n,

$$\mathbf{A}^n = \begin{pmatrix} 2^n & 3(2^n - 1) \\ 0 & 1 \end{pmatrix}.$$
 [5]

12	Prove, b	by	mathematical	induction,	that,	for	integers	$n \ge$	≥ 2,	,
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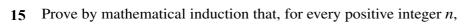
$$4^n > 2^n + 3^n. ag{5}$$

13 For the sequence  $u_1, u_2, u_3, \ldots$ , it is given that  $u_1 = 1$  and  $u_{r+1} = \frac{3u_r - 2}{4}$  for all r. Prove by mathematical induction that  $u_n = 4\left(\frac{3}{4}\right)^n - 2$ , for all positive integers n.

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14 Prove by mathematical induction that  $5^{2n} - 1$  is divisible by 8 for every positive integer n.

[5]



$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^x\sin x) = (\sqrt{2})^n \mathrm{e}^x\sin(x + \frac{1}{4}n\pi).$$
 [7]

16 Prove by mathematical induction that, for all non-negative integers n,

$$11^{2n} + 25^n + 22$$

is divisible by 24. [6]

It is given that  $\phi(n) = 5^n(4n+1) - 1$ , for  $n = 1, 2, 3, \dots$ . Prove, by mathematical induction, that  $\phi(n)$  is divisible by 8, for every positive integer n.

18 The sequence  $a_1, a_2, a_3, \dots$  is such that  $a_1 > 5$  and  $a_{n+1} = \frac{4a_n}{5} + \frac{5}{a_n}$  for every positive integer n.

Prove by mathematical induction that  $a_n > 5$  for every positive integer n. [5]

Prove also that  $a_n > a_{n+1}$  for every positive integer n. [2]

19 Prove by mathematical induction that, for all positive integers n,  $\sum_{r=1}^{n} \frac{1}{(2r)^2 - 1} = \frac{n}{2n+1}$ . [6]

State the value of 
$$\sum_{r=1}^{\infty} \frac{1}{(2r)^2 - 1}.$$
 [1]

20 Prove by mathematical induction that, for all positive integers n,  $10^n + 3 \times 4^{n+2} + 5$  is divisible by 9.

21 It is given that a diagonal of a polygon is a line joining two non-adjacent vertices. Prove, by

[6]

mathematical induction, that an *n*-sided polygon has  $\frac{1}{2}n(n-3)$  diagonals, where  $n \ge 3$ .

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22 Prove, by mathematical induction, that  $5^n + 3$  is divisible by 4 for all non-negative integers n.

[5]

23 Prove, by mathematical induction, that  $\sum_{r=1}^{n} r \ln\left(\frac{r+1}{r}\right) = \ln\left(\frac{(n+1)^n}{n!}\right)$  for all positive integers n. [6]

24	It is given that $f(n) = 2^{3n} + 8^{n-1}$ . By simplifying $f(k) + f(k+1)$ , or otherwise, prove by mathem	natical
	induction that $f(n)$ is divisible by 9 for every positive integer $n$ .	[6]

25 For the sequence  $u_1, u_2, u_3, ...$ , it is given that  $u_1 = 8$  and

$$u_{r+1} = \frac{5u_r - 3}{4}$$

for all r.

(i) Prove by mathematical induction that

$$u_n = 4\left(\frac{5}{4}\right)^n + 3,$$

for all positive integers n.

[5]

(ii) Deduce the set of values of x for which the infinite series

$$(u_1 - 3)x + (u_2 - 3)x^2 + \dots + (u_r - 3)x^r + \dots$$

is convergent. [2]

(iii) Use the result given in part (i) to find surds a and b such that

$$\sum_{n=1}^{N} \ln(u_n - 3) = N^2 \ln a + N \ln b.$$
 [3]

26 (i) Prove by mathematical induction that, for  $z \neq 1$  and all positive integers n,

$$1 + z + z^{2} + \dots + z^{n-1} = \frac{z^{n} - 1}{z - 1}.$$
 [5]

(ii) By letting  $z = \frac{1}{2}(\cos \theta + i \sin \theta)$ , use de Moivre's theorem to deduce that

$$\sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^m \sin m\theta = \frac{2\sin\theta}{5 - 4\cos\theta}.$$
 [5]

27	Prove by mathematical induction that $3^{3n} - 1$ is divisible by 13 for every positive integer $n$ .	[5]

**28** The sequence of real numbers  $a_1$ ,  $a_2$ ,  $a_3$ , ... is such that  $a_1 = 1$  and

$$a_{n+1} = \left(a_n + \frac{1}{a_n}\right)^{\lambda},$$

where  $\lambda$  is a constant greater than 1. Prove by mathematical induction that, for  $n \ge 2$ ,

$$a_n \geqslant 2^{g(n)},$$

where 
$$g(n) = \lambda^{n-1}$$
. [6]

Prove also that, for 
$$n \ge 2$$
,  $\frac{a_{n+1}}{a_n} > 2^{(\lambda-1)g(n)}$ . [3]

29 The sequence  $u_1$ ,  $u_2$ ,  $u_3$ , ... is such that  $u_1 = 1$  and

$$u_{n+1} = -1 + \sqrt{(u_n + 7)}.$$

- (i) Prove by induction that  $u_n < 2$  for all  $n \ge 1$ . [4]
- (ii) Show that if  $u_n = 2 \varepsilon$ , where  $\varepsilon$  is small, then

$$u_{n+1} \approx 2 - \frac{1}{6} \tag{2}$$

30	Prove by mathematical induction that, for all positive integers $n$ , $10^{3n} + 13^{n+1}$ is divisible by 7.	[5]

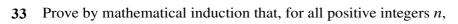
**31** Prove by induction that, for all  $n \ge 1$ ,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} (\mathrm{e}^{x^2}) = \mathrm{P}_n(x) \mathrm{e}^{x^2},$$

where  $P_n(x)$  is a polynomial in x of degree n with the coefficient of  $x^n$  equal to  $2^n$ . [6]

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32 Prove by mathematical induction that, for all non-negative integers n,  $7^{2n+1} + 5^{n+3}$  is divisible by 44.



$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^x\sin x) = 2^{\frac{1}{2}n}\mathrm{e}^x\sin\left(x + \frac{1}{4}n\pi\right).$$
 [7]

**34** Prove by mathematical induction that, for all positive integers n,

$$\frac{d^n}{dx^n} \left( \frac{1}{2x+3} \right) = (-1)^n \frac{n! \, 2^n}{(2x+3)^{n+1}}.$$
 [6]

- 35 Let  $I_n$  denote  $\int_0^\infty x^n e^{-2x} dx$ . Show that  $I_n = \frac{1}{2}nI_{n-1}$ , for  $n \ge 1$ . [2]
  - Prove by mathematical induction that, for all positive integers n,  $I_n = \frac{n!}{2^{n+1}}$ . [6]

36 Let  $S_N = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{N}{(N+1)!}$ . Prove by mathematical induction that, for all positive integers N,

$$S_N = 1 - \frac{1}{(N+1)!}. ag{5}$$

37 It is given that 
$$y = (1+x)^2 \ln(1+x)$$
. Find  $\frac{d^3y}{dx^3}$ . [3]

Prove by mathematical induction that, for every integer  $n \ge 3$ ,

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = (-1)^{n-1} \frac{2(n-3)!}{(1+x)^{n-2}}.$$
 [5]

**38** Prove by mathematical induction that, for every positive integer n,

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$
 [5]

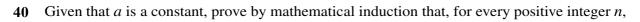
Express  $\sin^5 \theta$  in the form  $p \sin 5\theta + q \sin 3\theta + r \sin \theta$ , where p, q and r are rational numbers to be determined.

39 It is given that  $u_r = r \times r!$  for  $r = 1, 2, 3, \dots$  Let  $S_n = u_1 + u_2 + u_3 + \dots + u_n$ . Write down the values of

$$2! - S_1, \quad 3! - S_2, \quad 4! - S_3, \quad 5! - S_4.$$
 [2]

Conjecture a formula for  $S_n$ . [1]

Prove, by mathematical induction, a formula for  $S_n$ , for all positive integers n. [4]



$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x\mathrm{e}^{ax}) = na^{n-1}\mathrm{e}^{ax} + a^n x\mathrm{e}^{ax}.$$
 [6]

41 Using factorials, show that 
$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$
. [2]

Hence prove by mathematical induction that

$$(a+x)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}x + \dots + \binom{n}{r}a^{n-r}x^r + \dots + \binom{n}{n}x^n$$

for every positive integer n. [4]

**42** (i) Show that 
$$\frac{d^{n+1}}{dx^{n+1}}(x^{n+1}\ln x) = \frac{d^n}{dx^n}(x^n + (n+1)x^n\ln x).$$
 [2]

(ii) Prove by mathematical induction that, for all positive integers n,

$$\frac{d^n}{dx^n}(x^n \ln x) = n! \left( \ln x + 1 + \frac{1}{2} + \dots + \frac{1}{n} \right).$$
 [5]

43 The sequence of positive numbers  $u_1$ ,  $u_2$ ,  $u_3$ , ... is such that  $u_1 < 3$  and, for  $n \ge 1$ ,

$$u_{n+1} = \frac{4u_n + 9}{u_n + 4}.$$

(i) By considering  $3 - u_{n+1}$ , or otherwise, prove by mathematical induction that  $u_n < 3$  for all positive integers n. [5]

(ii) Show that  $u_{n+1} > u_n$  for  $n \ge 1$ .

[3]

44 It is given that  $y = e^x u$ , where u is a function of x. The rth derivatives  $\frac{d^r y}{dx^r}$  and  $\frac{d^r u}{dx^r}$  are denoted by  $y^{(r)}$  and  $u^{(r)}$  respectively. Prove by mathematical induction that, for all positive integers n,

$$y^{(n)} = e^{x} \left( \binom{n}{0} u + \binom{n}{1} u^{(1)} + \binom{n}{2} u^{(2)} + \dots + \binom{n}{r} u^{(r)} + \dots + \binom{n}{n} u^{(n)} \right).$$
 [8]

[You may use without proof the result  $\binom{k}{r} + \binom{k}{r-1} = \binom{k+1}{r}$ .]

45 It is given that  $y = \ln(ax + 1)$ , where a is a positive constant. Prove by mathematical induction that, for every positive integer n,

$$\frac{d^n y}{dx^n} = (-1)^{n-1} \frac{(n-1)! a^n}{(ax+1)^n}.$$
 [6]

46 Prove by mathematical induction that, for every positive integer n,

$$\frac{\mathrm{d}^{2n-1}}{\mathrm{d}x^{2n-1}}(x\sin x) = (-1)^{n-1} \left(x\cos x + (2n-1)\sin x\right).$$
 [7]

47	Prove by mathematical induction that $7^{2n} - 1$ is divisible by 12 for every positive integer $n$ .	[5]

48	The sequence $u_1, u_2, u_3, \dots$	. is such that $u_1$	$= 1$ and $u_{n+1}$	$=2u_n+1$ for $n \ge 1$ .

(a) Prove by induction that 
$$u_n = 2^n - 1$$
 for all positive integers  $n$ .

[5]

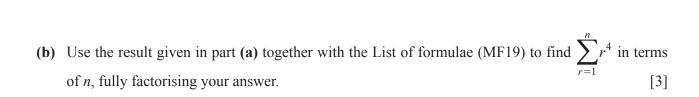
**(b)** Deduce that 
$$u_{2n}$$
 is divisible by  $u_n$  for  $n \ge 1$ . [2]

49	Prove by mathematical induction that $2^{4n} + 31^n - 2$ is divisible by 15 for all positive integers $n$ .	[6]

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50 (a) Prove by mathematical induction that, for all positive integers n,

$$\sum_{r=1}^{n} (5r^4 + r^2) = \frac{1}{2}n^2(n+1)^2(2n+1).$$



51 The sequence of positive numbers  $u_1$ ,  $u_2$ ,  $u_3$ , ... is such that  $u_1 > 4$  and, for  $n \ge 1$ ,

$$u_{n+1} = \frac{u_n^2 + u_n + 12}{2u_n}.$$

(a) By considering  $u_{n+1}-4$ , or otherwise, prove by mathematical induction that  $u_n > 4$  for all positive integers n. [5]

**(b)** Show that  $u_{n+1} < u_n$  for  $n \ge 1$ . [3]

- 52 Let  $\mathbf{A} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ , where a is a positive constant.
  - (a) State the type of the geometrical transformation in the x-y plane represented by **A**. [1]
  - **(b)** Prove by mathematical induction that, for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}.$$
 [5]

Let  $\mathbf{B} = \begin{pmatrix} b & b \\ a^{-1} & a^{-1} \end{pmatrix}$ , where b is a positive constant.

(c) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by  $\mathbf{A}^n\mathbf{B}$ .

53 The sequence of real numbers  $a_1$ ,  $a_2$ ,  $a_3$ , ... is such that  $a_1 = 1$  and

$$a_{n+1} = \left(a_n + \frac{1}{a_n}\right)^3.$$

(a) Prove by mathematical induction that  $\ln a_n \ge 3^{n-1} \ln 2$  for all integers  $n \ge 2$ . [6] [You may use the fact that  $\ln \left( x + \frac{1}{x} \right) > \ln x$  for x > 0.]

**(b)** Show that 
$$\ln a_{n+1} - \ln a_n > 3^{n-1} \ln 4$$
 for  $n \ge 2$ . [2]

54 It is given that  $y = xe^{ax}$ , where a is a constant.

Prove by mathematical induction that, for all positive integers n,

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = \left(a^n x + na^{n-1}\right) e^{ax}.$$
 [6]

	Prove by mathematical	1 : 1	C 11		. 0711 50 :-	1: :::1.1. 1. 40	F ( T
55	Prove by mathematical	l induction that.	. for all positive	integers $n$ . $\sqrt{2n}$	+9'/" - 50 is	divisible by 48.	16

56 The function f is such that f''(x) = f(x).

Prove by mathematical induction that, for every positive integer n,

$$\frac{\mathrm{d}^{2n-1}}{\mathrm{d}x^{2n-1}} (x f(x)) = x f'(x) + (2n-1) f(x).$$
 [7]